The Almost Invariant Subspace Problem

Caleb Suan

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Introduction

The notes that follow are a summary of the Undergraduate Reaseach Assistantship that I held during the Spring 2019 term under Professor Laurent Marcoux. The majority of the term was spent studying the Almost Invariant Subspace Problem, first introduced by Androulakis et al. in [1], and as such, these notes will closely follow the series of papers published after the original paper in 2009. Many of the proofs presented below are taken from their original sources, but with more details added to increase clarity.

In addition to the results of [1] [2] [3] [4] [5] [6] [7] [8] , we will make some comments of extending ideas to an almost invariant setting. While seeming promising at first, they quickly prove to have several issues which will be discussed in these notes. We will also note some questions of interest and some partial answers to them. The notes conclude with sections regarding notions of reflexivity and hyperreflexivity, following Conway [9].

These notes are subject to error both typographically and stylistically. I hope to fix any that I come across to improve the clarity of the material.

Throughout these notes, a subspace of a Banach space will always refer to a norm-closed subspace. Additionally, for a Banach space X, we will represent scalar operators of the form $\alpha I \in \mathcal{B}(X)$ simply by α.

Basic Ideas and Definitions

Definition 1.1. A subspace Y of a Banach space X is called a **half-space** if it is of both infinite dimension and infinite codimension in X.

Definition 1.2. If $T \in \mathcal{B}(X)$ and Y is a subspace of X, then Y is called **almost invariant under** T or T**-almost invariant**, if there exists a finite dimensional subspace F of X such that

$$TY \subseteq Y + F. \tag{1}$$

We call any subspace F satisfying Equation (1) an **error space of** Y **for** T. If Y is T-almost invariant, we call the minimal dimension of an error subspace the **defect of** Y **for** T. An error subspace F with dimension equal to the defect will be referred to as a **minimal error subspace**.

In contrast to the Invariant Subspace Problem which asks the question: *Does every operator on an infinite dimensional Banach space have an invariant subspace?* The Almost Invariant Subspace Problem asks: *Does every operator on an infinite dimensional Banach space have an almost invariant half-space?*. We note that only half-spaces are considered here as a result of the following lemma.

Lemma 1.3. *Let X be a Banach space and $T \in \mathcal{B}(X)$. If Y is a subspace of X which is not a half-space, then Y is T-almost invariant.*

Proof. Since Y is not a half-space, it either is finite dimensional or finite codimensional in X.

Suppose Y is finite dimensional, then so is TY, and $TY \subseteq Y + TY$.

If Y is finite codimensional in X, then there exists a finite dimensional subspace Z such that $X = Y \oplus Z$. Clearly we have $TY \subseteq X = Y + Z$. ☐

We note a relationship between almost invariant half-spaces and invariant half-spaces modulo a finite rank perturbation.

Proposition 1.4. *Let $T \in \mathcal{B}(X)$ and $Y \subseteq X$ be a half-space. Y is T-almost invariant if and only if Y is $T+K$-invariant for some finite rank operator K.*

Proof. Suppose that Y is T-almost invariant. Let F be a minimal error subspace of Y for T. We first show that the sub $Y + F$ is direct.

Suppose that there exists some $x \in Y \bigcap F$. Since F is finite dimensional, we may extend x to a basis $\{x, x_2, \cdots, x_n\}$ of F. Let $F_0 = \operatorname{span}\{x_2, \cdots, x_n\}$. It is clear that $TY \subseteq Y + F_0$, contradicting the minimality of $\dim F$.

Let P be the projection

$$\begin{aligned} P \colon Y \oplus F &\to F \\ y + f &\mapsto f. \end{aligned}$$

It is clear that P is a finite rank operator. With a choice of basis for F and repeated applications of the Hahn-Banach Theorem, we may extend P such that it acts on all of X. This extension, $\tilde{P}$ is also a finite rank operator and satisfies $\tilde{P}|_{Y+F} = P$.

Define $K = -\tilde{P}T$. K has finite rank since $\tilde{P}$ does and for $y \in Y$, we have that for some $y' \in Y$ and $f \in F$,

$$(T + K)y = Ty - \tilde{P}Ty = y' + f - \tilde{P}(y' + f) = y' + f - f = y' \tag{2}$$

Hence Y is $T + K$-invariant for some finite rank operator K.

To see the converse, we note that if Y is $T + K$-invariant, then $(T + K)Y \subseteq Y$. It follows that $TY \subseteq Y + KY$. Since KY is finite dimensional, Y is T-almost invariant. □

An important result is that if an operator has an almost invariant half-space, then so does its adjoint. To show this, we require a couple of lemmas.

Lemma 1.5. *Let X be a Banach space and $Y \subseteq X$ be a subspace. Then Y is infinite codimensional in X if and only if $Y^\perp$ is infinite dimensional. In particular, Y is a half-space if and only if both Y and $Y^\perp$ are infinite dimensnional.*

Lemma 1.6. *Let X be a Banach space and $Y \subseteq X$ be a subspace. Y is a half-space if and only if $Y^\perp$ is a half-space (in X^*).*

Proof. Suppose that Y is a half-space, by Lemma 1.5, $Y^\perp$ is infinite dimensional. Consider the isomentric embedding $\iota \colon X \to X^{**}$. We see $\iota Y \subseteq (Y^\perp)^\perp$, so $(Y^\perp)^\perp$ is also infinite dimensional. Lemma 1.5 shows that $Y^\perp$ is a half-space.

Suppose instead that $Y^\perp$ is a half-space, then $Y^\perp$ is infinite codimensional. It follows that Y must also be infinite dimensional. To see this, note that given an m-dimensional subspace K of X with basis $\{x_1, \cdots, x_m\}$, by the Hahn-Banach Theorem, we may construct linear functionals $x_1^*, \cdots, x_m^*$ such that $x_i^*(x_j) = \delta_{ij}$. Let $L = \text{span}\{x_1^*, \cdots, x_m^*\}$, we will show that $X^* = K^\perp \oplus L$.

First, it is clear that $K^\perp \bigcap L = \{0\}$ as each x_i^* does not annihilate K. For $x^* \in X^*$, we can take $y^* = \sum_{i=1}^m x^*(x_i)x_i^* \in L$. We see $x^* - y^* \in K^\perp$, showing the contrapositive.

Lemma 1.5 shows that Y is also infinite codimensional, and hence Y is a half-space. □

The previous lemmas now allows us to show that if $T \in \mathcal{B}(X)$ has an almost invariant half-space, then so does T^*.

Proposition 1.7. *Let $T \in \mathcal{B}(X)$. If T has an almost invariant half-space with defect k, then so does T^*.*

Proof. Let Y be a T-almost invariant half-space and let F be a minimal error space. By the proof of Proposition 1.4, the sum $Y + F$ is direct. Since F is finite dimensional, we can consider its complement W satisfying $X = W \oplus F$. We see that $Y \subseteq W$ and that $W^\perp$ is finite dimensional.

Consider $(Y+F)^\perp \subseteq X^*$. Since F is finite dimensional, $Y+F$ and $(Y+F)^\perp$ are also half-spaces. Given $z^* \in (Y+F)^\perp$ and $y \in Y$, we have $T^*z^*(y) = z^*(Ty) = 0$ since $Ty \in Y+F$. It follows that $T^*(Y+F)^\perp \subseteq Y^\perp$. We want to show that T^*-almost invariance of $(Y+F)^\perp$ and it suffices to show that $Y^\perp = (Y+F)^\perp + W^\perp$.

It is clear that $(Y+F)^\perp \subseteq Y^\perp$. Also, $Y \subseteq W$ implies that $W^\perp \subseteq Y^\perp$ and so $(Y+F)^\perp + W^\perp \subseteq Y^\perp$.

For the other containment, we consider a basis $\{f_1, \cdots, f_m\}$ of F and biorthogional functionals $\{f_1^*, \cdots, f_m^*\}$ in $W^\perp$. Such functionals may be constructed by the Hahn-Banach Theorem. Each $f_i^* \in Y^\perp$ and so for $x^* \in Y^\perp$, we have $x^* - \sum_{i=1}^m x^*(f_i)f_i^* \in (Y+F)^\perp$. This shows that $x^* \in (Y_F)^\perp + W^\perp$, completing the proof. □

We note that if we consider a Hilbert space $\mathcal{H}$ instead, that Proposition 1.7 follows by taking $Y^\perp$ to be the T^*-almost invariant half-space.

Proofs for Certain Classes of Operators

For an operator $T \in \mathcal{B}(X)$, a non-zero vector $e \in X$, and $\lambda \in \rho(T)$, we may define a vector $h(\lambda, e) \in X$ by

$$h(\lambda, e) = (\lambda - T)^{-1}e.$$

If $|\lambda| > \operatorname{spr}(T)$, then the power series

$$h(\lambda, e) = \sum_{n=0}^{\infty} \lambda^{-n-1} T^n e. \tag{3}$$

Note that $e = (\lambda - T)h(\lambda, e)$, rearranging this yields

$$Th(\lambda, e) = \lambda h(\lambda, e) - e. \tag{4}$$

Equation (4) gives the following result.

Lemma 2.1. *Let X be a Banach space, $T \in \mathcal{B}(X)$, $0 \neq e \in X$ and $A \subseteq \rho(T)$. Set*

$$Y = \overline{\operatorname{span}}\{h(\lambda, e) \colon \lambda \in A\}. \tag{5}$$

Then Y is a T-almost invariant subspace with $TY \subseteq Y + \operatorname{span}\{e\}$.

We would like to use Lemma 2.1 to construct almost invariant half-spaces for certain operators. The first set of operators we work on include quasinilpotent weighted shifts.

Definition 2.2. A sequence $\{x_n\}_n$ in a Banach space X is called minimal if $x_k \notin [x_n]_{n \neq k}$ for each k.

Lemma 2.3. *A sequence $\{x_n\}_n$ in X is minimal if and only if there exists a sequence $\{x_n^*\}$ of biorthogonal functionals in X^*.*

Proof. Suppose $\{x_n\}_n$ is minimal. For each k, we can define the subspace $E_k = [x_n]_{n \neq k}$. By minimality, each E_k does not contain x_k. By the Hahn-Banach Theorem, we may construct the desired biorthogonal functionals $\{x_n^*\}$.

Conversely, fix k, and consider some $z \in F_k = \operatorname{span}\{x_n \colon n \neq k\}$. We may write $z = \sum_{j=1}^{m} c_{n_j} x_{n_j}$, then $x_k^*(z) = 0$. By the continuity of x_k^*, $x_k^*(x) = 0$ for all $x \in \overline{F_k}$. By definition, we have $x_k^*(x_k) = 1$, so $x_k \notin \overline{F_k}$ and hence the sequence $\{x_n\}_n$ is minimal. □

The following results will be important in showing the existence of almost invariant half-spaces of the operators.

Lemma 2.4. *Given a sequence $\{r_n\}_n$ of positive real numbers, there exists a sequence $\{c_n\}_n$ of positive real numbers such that the series $\sum_{n=0}^{\infty} c_n r_{n+k}$ converges for each k.*

Proof. For each n, take $c_n = \frac{1}{2^n} \min\left\{\frac{1}{r_0}, \cdots, \frac{1}{r_{2n}}\right\}$. If $n \geqslant k$, then $n + k \leqslant 2n$, and hence $c_n r_{n+k} \leqslant \frac{1}{2^n}$. It follows that

$$\sum_{n=0}^{\infty} c_n r_{n+k} \leqslant \sum_{n=0}^{k-1} c_n r_{n+k} + \sum_{n=k}^{\infty} \frac{1}{2^n} < \infty. \tag{6}$$

□

Theorem 2.5 (Great Picard Theorem)**.** *If f is a complex-analytic function with an essential singularity at z, then on any punctured neighbourhood of z, f takes on all complex values, except at most one, infinitely often.*

Using these results, we may show the existence of almost invariant half-spaces for certain classes of operators.

Theorem 2.6. *Let X be a Banach space and $T \in \mathcal{B}(X)$. Suppose that T satisfies*

a) the unbounded component of $\rho(T)$ contains a punctured open disk centered at 0;

b) there exists a vector $e \in X$ whose orbit $\{T^n e\}_n$ is a minimal sequence.

Then T has an almost invariant half-space.

Proof. Since $\{T^n e\}_n$ is a minimal sequence, it follows that for any non-zero polynomial $p \in \mathbb{C}[z]$, we must have $p(T)e \neq 0$. This also shows that the sequence $\{T^n e\}_n$ is linearly independent.

Define a subspace Y as in Lemma 2.1. We want to show that $\{h(\lambda, e) \colon \lambda \in A\}$ is linearly independent. This will show that Y is infinite dimensional by choosing A to be a infinite subset of $\rho(T)$.

Suppose there exist non-zero scalars $a_1, \cdots, a_m$ and distinct $\lambda_1, \cdots, \lambda_m$ in A such that

$$a_1 h(\lambda_1, e) + \cdots + a_n h(\lambda_n, e) = 0. \tag{7}$$

By applying the operator $(\lambda_1 - T) \cdots (\lambda_m - T)$ to both sides of Equation (7), we get $p(T)e = 0$ for some non-zero polynomial p, contradicting our earlier observation.

Set $x_n = T^n e$. Since $\{x_n\}_n$ is minimal, the biorthogonal functionals $\{x_n^*\}_n$ are bounded (and non-zero). Let $r_n = \|x_n^*\|$. By Lemma 2.4, there exists a sequence $\{c_n\}_n$ of positive real numbers such that $b_k = \sum_{n=0}^{\infty} c_n r_{n+k} < \infty$ for each k. Without loss of generality, we may assume that $\sqrt[n]{c_n} \to 0$ as shrinking the values of the c_n's will not affect convergence of the series.

Consider the function $F \colon \mathbb{C} \to \mathbb{C}$ defined by $F(z) = \sum_{n=0}^{\infty} c_n z^n$. By the Cauchy-Hadamard Theorem, F is entire. Set $H(z) = F(\frac{1}{z})$. H has an essential singularity at $z = 0$.

By the Great Picard Theorem (Theorem 2.5), we have that in a punctured neighbourhood of 0, $H(z)$ achieves all complex values, except at most one, infinitely often, hence so does $F(z)$. By picking an appropriate $d < 0$, and replacing c_0 with $c_0 - d$, we may assume without loss of generality that the set $Z(F) = \{z \in \mathbb{C} \colon F(z) = 0\}$ is infinite.

Since $Z(F)$ is infinite, we can pick a sequence $\{\lambda_n^{-1}\}_n$ of distinct elements in $Z(F)$. Moreover, since F is non-constant, $Z(F)$ has no accumulation points, and hence $|\lambda_n| \to 0$. By condition a), we may assume that each λ_n lies in the unbounded component of $\rho(T)$ so $h(\lambda, e)$ is defined for each n.

Set $Y = [h(\lambda_n, e)]_{n \geqslant 0}$. Earlier observations show that Y is T-almost invariant and infinite dimensional. We show that $Y^\perp$ is also infinite dimensional by constructing a sequence of linearly independent functionals $\{f_n\}_n$ such that each f_n is in $Y^\perp$.

For each k, set $F_k(z) = z^k F(z)$. In terms of Taylor series expansions, we see that

$$F_k(z) = z^k F(z) = z^k \sum_{n=0}^{\infty} c_n z^n = \sum_{n=0}^{\infty} c_n z^{n+k} = \sum_{n=0}^{\infty} c_n^{(k)} z^n. \tag{8}$$

where we define the constants $c_n^{(k)}$ by

$$c_n^{(k)} = \begin{cases} 0 & \text{if } n < k. \\ c_{n-k} & \text{if } n \geqslant k. \end{cases} \tag{9}$$

For each k, we define a function on $\operatorname{span}\{x_n\}$ by $f_k(x_n) = c_n^{(k)}$. This functional is well-defined since the sequence $\{x_n\}_n$ is minimal. We wish to show that each f_k is bounded. Let $x \in \operatorname{span}\{x_n\}$. We may write $x = \sum_{n=0}^{m} x_n^*(x) x_n$ for some $m \geqslant 0$. This gives

$$\begin{aligned} |f_k(x)| &= \left| f_k\left(\sum_{n=0}^{\infty} x_n^*(x) x_n \right) \right| \\ &\leqslant \left(\sum_{n=0}^{m} \|x_n^*\| c_n^{(k)} \right) \|x\| \\ &= \left(\sum_{n=k}^{m} r_n c_{n-k} \right) \|x\| \\ &\leqslant \left(\sum_{n=k}^{\infty} r_n c_{n-k} \right) \|x\| \\ &= b_k \|x\|. \end{aligned} \tag{10}$$

Thus each f_k is bounded. By the Hahn-Banach Theorem, we may extend its domain to all of X. Now we show that f_k annihilates Y.

Fix k. If $|\lambda| > \mathrm{spr}(T)$, we have

$$\begin{aligned} f_k(h(\lambda, e)) &= f_k\left(\sum_{n=0}^{\infty} \lambda^{-n-1} x_n\right) \\ &= \lambda^{-1} \sum_{n=0}^{\infty} \lambda^{-n} c_n^{(k)} \\ &= \lambda^{-1} F_k(\lambda^{-1}) \\ &= \lambda^{-k-1} F(\lambda^{-1}). \end{aligned} \tag{11}$$

Analyticity of the maps $\lambda \mapsto f_k(h(\lambda, e))$ and $\lambda \mapsto \lambda^{-k-1}F(\lambda^{-1}$ on $\rho(T)\setminus\{0\}$ and the Uniqueness Property show that the maps agree on that domain. Since each $\lambda_n^{-1} \in Z(F)$, f_k annihilates Y.

Since $f_k(x_n) = c_n^{(k)} \neq 0$ for $n \geqslant k$, each f_k is non-zero. Suppose that they were dependent, then we may write

$$f_l = \sum_{n=m}^{l-1} a_n f_n.$$

with $a_m \neq 0$. Evaluating both sides of the equation at x_m gives

$$f_l(x_m) = c_m^{(l)} = 0 \neq a_m c_0 = \sum_{n=m}^{l-1} a_n c_m^{(n)} = \sum_{n=m}^{l-1} a_n f_n(x_m). \tag{12}$$

This is a contradiction. Hence the f_k are linearly independent and Y is a T-almost invariant half-space. □

Corollary 2.7. *If T is a weighted forward shift with weights converging to 0, then T has an almost invariant half-space.*

Proof. T is quasinilpotent, so it satisfies condition a) of Theorem 2.6. Also, the orbit of e_0 forms a minimal sequence. □

Remark 2.8. Condition a) in Theorem 2.6 may be replaced with a weaker condition, requiring only an open segment of a punctured disk to be contained in $\rho(T)$.

In light of Theorem 2.6, by considering the Hilbert spaces instead of general Banach spaces, one might wish to define an almost reducing half-space of an operator T in an analogous manner. A natural question to ask is the following: *Does every operator acting on a Hilbert space have an almost reducing half-space?*

Remark 2.9. To check if almost reducing half-spaces are worth exploring, one would like to be able to exhibit the existence of one. In particular, one would like to know if the construction in the proof of Theroem 2.6 produces an almost reducing half-space. As an example, we could consider a Donoghue shift $D \in \mathcal{B}(\ell_2(\mathbb{N}))$ with weight sequence $\{w_n\}_n$. Recall that D is defined by

$$De_0 = 0, \quad De_i = w_i e_{i-1}, \quad i \in \mathbb{N}.$$

Let $T \in \mathcal{B}(\ell_2(\mathbb{N}))$ be the diagonal operator defined by

$$Te_i = w_{i+1} e_i, \quad i \in \mathbb{N} \cup \{0\}.$$

For a vector $e \in \ell_2(\mathbb{N})$, and $\lambda \neq 0$, one may calculate

$$D(\lambda - D^*)^{-1} e = \frac{1}{\lambda} De + \frac{1}{\lambda} T^2 (\lambda - D^*)^{-1} e. \tag{13}$$

Hence we see that the subspace $Y_e = \overline{\text{span}}\{(\lambda - D^*)^{-1} e : \lambda \in A\}$ for some subset $A \in \rho(D)$ is D-almost invariant if and only if it is T^2-almost invariant.

At the time of writing, I am unsure if Y_e is D-almost reducing for any vector e with $\left\{(D^*)^n e\right\}_n$ being a minimal sequence.

The existence of almost invariant half-spaces can be extended to another class of operators. The proof of which follows similarly to that of Theorem 2.6 but uses a Blaschke product instead of an entire function.

Theorem 2.10. *Let X be a Banach space and $T \in \mathcal{B}(X)$. Suppose that T satisfies*

a) $\operatorname{spr}(T) \leqslant 1$*;*

b) There exists a vector $e \in X$ whose orbit $\{T^n e\}_n = \{x_n\}_n$ is a minimal sequence and

$$\sum_{n=1}^{\infty} \frac{\|x_n^*\|}{n} < \infty. \tag{14}$$

Then T has an almost invariant half-space.

Proof. Let $\mathbb{D}$ be the open unit disk in $\mathbb{C}$. Given a sequence $\{\lambda_n\}_n$ in $\mathbb{D}$ such that $\sum_{n=1}^{\infty}(1 - |\lambda_n|) < \infty$, the corresponding Blaschke product is defined by

$$B(z) = \prod_{n=1}^{\infty} \frac{|\lambda_n|}{\lambda_n} \frac{\lambda_n - z}{1 - \overline{\lambda_n} z}. \tag{15}$$

B is a bounded analytic function on $\mathbb{D}$ with zeros exactly at the λ_n.

A result in [10] shows the existence of a sequence $\{\lambda_n\}_n$ in $\mathbb{D}$ such that the Taylor coefficients a_n of its infinite Blaschke product B are of the order $O\left(\frac{1}{n+1}\right)$ and so $B^{(n)}(0) = O\left(\frac{n!}{n+1}\right)$.

For $k \geqslant 1$, define the functions $F_k(z) = z^k B(z)$. These functions F_k are linearly independent as linear dependence would imply $p(z)B(z) = 0$ on $\mathbb{D}$ for some polynomial $p(z) \in \mathbb{C}[z]$. However, the zeros of $p(z)B(z)$ are countable, giving a contradiction.

We have that

$$F_k(z) = z^k B(z) = z^k \sum_{n=0}^{\infty} \frac{B^{(n)}(0)}{n!} z^n = \sum_{n=m}^{\infty} \frac{B^{(n-m)}(0)}{(n-m)!} z^n \tag{16}$$

and so

$$F_k^{(n)}(0) = \begin{cases} 0 & \text{if } n < m. \\ \frac{n!}{(n-m)!} B^{(n-m)}(0) \leqslant C \frac{n!}{n-m+1} & \text{if } n \geqslant m. \end{cases} \tag{17}$$

Since each $\lambda_n \in \mathbb{D}$ and $\operatorname{spr}(T) \leqslant 1$, it follows that $\lambda_n^{-1} \in \rho(T)$ for each n and so $h(\lambda_n^{-1}, e)$ is well-defined. Set $Y = [h(\lambda_n^{-1}, e)]_{n \geqslant 0}$. As seen before, Y is infinite dimensional and T-almost invariant.

Define linear functionals f_k on $\operatorname{span}\{x_n\}$ by $f_k(x_n) = \frac{F_k^{(n)}(0)}{n!}$. Since the x_n are linearly independent, f_k is well-defined for each k. To show that each f_k is bounded, take $x \in \operatorname{span}\{x_n\}$. We may write $x = \sum_n \alpha_n x_n$ and

$$\begin{aligned} |f_k(x)| &= \left| \sum_n \alpha_n \frac{F_k^{(n)}(0)}{n!} \right| \\ &\leqslant C \sum_{n \geqslant k} \frac{|\alpha_n|}{n-k+1} \\ &= C \sum_{n \geqslant k} \frac{|x_n^*(x)|}{n-k+1} \\ &\leqslant C\|x\| \sum_{n \geqslant k} \frac{\|x_n^*\|}{n-k+1}. \end{aligned} \tag{18}$$

If $k = 0$ or 1, the series in the Equation (18) converges by our assumptions. Otherwise, note that when $n \geqslant k$ and $k \geqslant 1$,

$$k(n-k+1) = (k-1)(n-k) + n \geqslant n. \tag{19}$$

Combining Equations (18) and (19), we have

$$\sum_{n \geqslant k} \frac{\|x_n^*\|}{n-k+1} = k \sum_{n \geqslant k} \frac{\|x_n^*\|}{k(n-k+1)} \leqslant k \sum_{n \geqslant k} \frac{\|x_n^*\|}{n} < \infty. \tag{20}$$

Hence each f_k is bounded. By the Hahn-Banach Theorem, we may extend the domain of f_k to all of X.

If $|\lambda| < 1$, we have

$$f_k(h(\lambda^{-1}, e)) = f_k\big(\lambda \sum_{n=0}^{\infty} \lambda^n x^n\big) = \lambda \sum_{n=0}^{\infty} \lambda^n \frac{F_k^{(n)}(0)}{n!} = \lambda F_k(\lambda). \tag{21}$$

By analyticity, these functions agree on $\mathbb{D}\setminus\{0\}$. Since each λ_n is a zero of B and hence F_k, it follows that the f_k annihilate Y.

Suppose that the f_k are linearly independent. Then for some $m > 0$ and $\alpha_0, \cdots, \alpha_m$ with $\alpha_m \neq 0$, we have

$$\alpha_0 f_0 + \cdots + \alpha_m f_m = 0.$$

Consider the Taylor series expansions of the F_k and note

$$\begin{aligned}
\alpha_0 F_0 + \cdots + \alpha_m F_m &= \alpha_0 \sum_{n=0}^{\infty} \frac{F_0^{(n)}(0)}{n!} z^n + \cdots + \alpha_m \sum_{n=0}^{\infty} \frac{F_m^{(n)}(0)}{n!} z^n \\
&= \sum_{n=0}^{\infty} \frac{\alpha_0 F_0^{(n)}(0) + \cdots + \alpha_m F_m^{(n)}(0)}{n!} z^n \\
&= \sum_{n=0}^{\infty} \frac{\alpha_0 f_0 + \cdots + \alpha_m f_m}{n!} z^n \\
&= 0.
\end{aligned} \tag{22}$$

This contradicts the earlier observation that the functions F_k were linearly independent. □

The conditions set on the operator T in Theorem 2.10 seem rather restrictive. As such, one would want examples of an operator satisfying Theorem 2.10 and not Theorem 2.6. Below is an example of such an operator.

Example 2.11. Let W be a weighted forward shift with non-increasing non-zero weights $\{w_n\}_n$. W acts on the basis $\{e_0, e_1, \cdots\}$ by $We_n = w_{n+1}e_{n+1}$. Then $\|W^n\| = \prod_{k=1}^{n} |w_n|$. By the Spectral Radius Forumla,

$$\operatorname{spr}(W) = \lim_{n\to\infty} \|W^n\|^{\frac{1}{n}} = \lim_{n\to\infty} \left(\prod_{k=1}^{n} |w_k| \right)^{\frac{1}{n}}. \tag{23}$$

Since W is a forward shift, we see that the orbit of e_0 is a minimal sequence. Let $x_n = W^n e_0$, then

$$\|x_n^*\| = \prod_{k=1}^{n} \frac{1}{|w_k|}. \tag{24}$$

For W to satisfy the conditions of Theorem 2.10, we require that

$$\operatorname{spr}(W) = \lim_{n\to\infty} \left(\prod_{k=1}^{n} |w_k| \right)^{\frac{1}{n}} \leqslant 1 \tag{25}$$

and

$$\lim_{n\to\infty} \sum_{k=1}^{n} \frac{\|x_k^*\|}{k} = \lim_{n\to\infty} \sum_{k=1}^{n} \frac{1}{k} \left(\prod_{j=1}^{k} \frac{1}{|w_j|} \right) < \infty \tag{26}$$

We note that finding a monotone decreasing sequence $\{p_n\}_n$ of non-negative real numbers satisfying

$$\lim_{n\to\infty}\left(\prod_{k=1}^{n}(1+p_k)\right)^{\frac{1}{n}} \leqslant 1 \tag{27}$$

and

$$\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{k}\left(\prod_{j=1}^{k}\frac{1}{1+p_j}\right) < \infty \tag{28}$$

will suffice in providing an example. Indeed, one can see this by picking the weight sequence for W to be $\{w_n\}_n = \{1+p_n\}_n$.

Consider the sequence $p_1 = 2, p_k = \frac{1}{k-1}$ for $k \geqslant 2$. For $k \in \mathbb{N}$,

$$\prod_{j=1}^{k}\frac{1}{1+p_j} = \frac{1}{3}\prod_{j=2}^{k}\frac{j-1}{j} = \frac{1}{3k}. \tag{29}$$

Substituting Equation (29) yields

$$\sum_{k=1}^{n}\frac{1}{k}\left(\prod_{j=1}^{k}\frac{1}{1+p_j}\right) = \sum_{k=1}^{n}\frac{1}{3k^2} \to \frac{\pi^2}{18}. \tag{30}$$

This satisfies the second condition (Equation (28)).

To check the first condition, we note the following inequalities: for each $n \in \mathbb{N}$,

$$\begin{aligned} 1 &\leqslant \prod_{k=1}^{n}(1+p_k) \\ &\leqslant \exp\left(\sum_{k=1}^{n}p_k\right) \\ &\leqslant \exp\left(3+\sum_{k=1}^{n-1}\frac{1}{k}\right) \\ &\leqslant \exp(3+\log(n-1)) \\ &= e^3(n-1). \end{aligned} \tag{31}$$

From Equation (31), we get

$$1^{\frac{1}{n}} \leqslant \left(\prod_{k=1}^{n}(1+p_k)\right)^{\frac{1}{n}} \leqslant (e^3(n-1))^{\frac{1}{n}}. \tag{32}$$

The limits of the left and right sides of Equation (32) as n tends to infinity are both 1. This shows the first condition (Equation (27)).

The operator W constructed above does not satisfy the conditions of Theorem 2.6 as the unbounded component of its resolvent does not contain a punctured disk centered at 0. Indeed, its spectral radius is 1, so its spectrum contains some non-zero point. Further, since W is a weighted shift with positive real weights, its spectrum has rotational symmetry about 0, and hence contains some circle with positive radius centered at 0.

A Scalar-Plus-Finite-Rank Characterization

We first note the following characterization of scalar operators acting on a Banch space:

Proposition 3.1 (Scalar Characterization)**.** *Let X be a Banach space and $T \in \mathcal{B}(X)$. Every subspace is invariant under T if and only if $T = \alpha$ for some $\alpha \in \mathbb{C}$.*

We now work to prove an analog in the almost invariant setting. In particular, we wish to prove the following:

Proposition 3.2 (Scalar-Plus-Finite-Rank Characterization)**.** *Let X be a Banach space and $T \in \mathcal{B}(X)$. Every subspace is almost invariant under T if and only if $T = \alpha + F$ for some $\alpha \in \mathbb{C}$ and some finite rank operator F.*

Before proving the proposition in the general Banach space setting, we first look at the case where instead we have a separable Hilbert space $\mathcal{H}$. We give a proof that heavily uses the fact that we are working in a Hilbert space. An alternate proof of the following proposition is given by Marcoux, Popov, and Radjavi in [3].

Proposition 3.3 (Scalar-Plus-Finite-Rank Characterization (Hilbert Spaces))**.** *Let $\mathcal{H}$ be a separable Hilbert space and $T \in \mathcal{B}(\mathcal{H})$. Every subspace is almost invariant under T if and only if $T = \alpha + F$ for some $\alpha \in \mathbb{C}$ and some finite rank operator F.*

Proof. Suppose that every subspace is T-almost invariant. Given any subspace $Y \subseteq \mathcal{H}$, under the decomposition $\mathcal{H} = Y \oplus Y^{\perp}$, we get that

$$T = \begin{bmatrix} T_1 & F_2 \\ F_1 & T_2 \end{bmatrix}$$

where F_1 and F_2 are finite rank.

If π is the canonical map from $\mathcal{B}(\mathcal{H})$ to the Calkin algebra $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$, then we see that

$$\pi(T) = \begin{bmatrix} \pi(T_1) & 0 \\ 0 & \pi(T_2) \end{bmatrix}.$$

It follows that T is essentially reductive, since this holds for any decomposition $\mathcal{H} = Y \oplus Y^{\perp}$. A result of Moore [11] shows that T is essentially normal. Further, combining this with results of Harrison [12] tells us that T has Lavrientiev essential spectrum and thus that $T = N + K$ for some normal operator N and some compact operator K.

Let $M \subseteq \mathcal{H}$ be a half-space and P be the projection with range M. Under the decomposition $\mathcal{H} = M \oplus M^{\perp}$, P looks like

$$P = \begin{bmatrix} I & 0 \\ 0 & 0 \end{bmatrix}.$$

Since M is a half-space, we get that $p = \pi(P) \in \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ and that $0 \neq p \neq 1$. Furthermore, P is a projection, so $P = P^* = P^2$, and $p = p^* = p^2$. We will show that every projection in the Calkin algebra is the image of a projection in $\mathcal{B}(\mathcal{H})$.

Suppose $q \in \mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is a projection. If $0 \neq q \neq 1$, then by the Spectral Mapping Theorem, we must have that $\sigma(q) = \{0, 1\}$. Let $Q_0 \in \mathcal{B}(\mathcal{H})$ be such that $\pi(Q_0) = q$. Such a Q_0 exists since π is surjective. Set $Q_1 = \frac{Q_0 + Q_0^*}{2}$. Q_1 is self-adjoint and $\pi(Q_1) = q$.

Since Q_1 is self-adjoint, it is normal and we may apply the Spectral Theorem for Normal Operators. We have that $\sigma_e(Q_1) = \sigma(q) = \{0, 1\}$, and since Q_1 is normal, we get $\sigma(Q_1) \backslash \sigma_e(Q_1)$ is the set of isolated eigenvalues of Q_1 of finite multiplicity. These eigenvalues converge to either 0 or 1. By approximating using the finite rank operators corresponding to the eigenspaces, we can construct a compact operator K_1 such that $Q = Q_1 + K$ has spectrum $\sigma(Q) = \{0, 1\}$ and is self-adjoint. From this, it follows that $Q = Q^* = Q^2$ by the continuous functional calculus since the function $f(z) = z$ satisfies $f = \overline{f} = f^2$ on $\sigma(Q) = \{0, 1\}$.

This shows the existence of a projection $Q \in \mathcal{B}(\mathcal{H})$ such that $\pi(Q) = q$ since compact perturbations do not change images in the Calkin algebra. Hence every projection in $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$ is the image of a projection in $\mathcal{B}(\mathcal{H})$.

Since every half-space is almost invariant under T, for any projection Q onto a half-space M, we get that both $(I - Q)TQ$ and $QT(I - Q)$ are finite rank. If $q = \pi(Q)$ and $t = \pi(T)$, then $(1 - q)tq = 0 = qt(1 - q)$ and so $qt = tq$. Thus t commutes with every projection in $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$.

Given an arbitrary $S \in \mathcal{B}(H)$, we may split S into its real and imaginary parts. Using the Spectral Theorem for Normal Operators and applying it to the real and imaginary parts of S, one can see that the span of the projections are dense in $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$. By the continuity of multiplication, it follows that t commutes with every element in $\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H})$, so $t = \pi(T) \in Z(\mathcal{B}(\mathcal{H})/\mathcal{K}(\mathcal{H}))$. Since the center of the Calkin algebra consists of just the scalars, it follows that $\pi(T) = \alpha + \mathcal{K}(\mathcal{H})$ and $T = \alpha + K$ for some $\alpha \in \mathbb{C}$ and some compact operator K. We wish to show that this compact operator K is indeed finite rank.

Adding a scalar multiple of the identity does not change the invariant and almost invariant subspaces of an operator, so replacing T with $T-\alpha$, we see that every half-space is almost invariant under K.

K^*K is a compact, self-adjoint (hence normal) operator. Thus we can apply the Spectral Theorem for Compact Normal Operators. The eigenvalues of K^*K form a countable (but possibly finite) sequence $\{\lambda_n\}_n$ converging to 0 and the corresponding eigenspaces are finite dimensional.

Since every subspace is almost invariant under K, the same is true for K^* and K^*K. Suppose that the sequence $\{\lambda_n\}_n$ were countably infinite. For each eigenvalue λ_n, we map pick a corresponding eigenvector u_n. Since eigenspaces for distinct eigenvalues are mutually orthogonal, the u_n are linearly independent.

Consider the vectors $\{u_{4n} + u_{4n+1}\}$ and their closed span $Y = [u_{4n} + u_{4n+1}]_{n\geqslant 0}$. It is clear that this subspace is infinite dimensional and infinite codimensional, so Y is a half-space. Furthermore, by construction, Y is not almost invariant for K^*K, a contradiction. So the sequence $\{\lambda_n\}_n$ must be finite.

It follows from each eigenspace being finite dimensional that K^*K is a finite rank operator. This gives that $K^*|_{\overline{\operatorname{ran} K}}$ is finite rank. As $\overline{\operatorname{ran} K} = (\ker K^*)^\perp$, we have that the map

$$K^*|_{\overline{\operatorname{ran} K}} = K^*|_{(\ker K^*)^\perp} \colon (\ker K^*)^\perp = \overline{\operatorname{ran} K} \to \operatorname{ran} K^*$$

is bijective. Hence $\operatorname{ran} K$ is finite dimensional and K is finite rank.

The converse is readily apparent. □

A proof of the general case (Proposition 3.2) was given by Assadi, Farzaneh, and Mohammadinejad in [8]. We follow the method that they used to show this characterization.

Lemma 3.4. *Let X be a Banach space and $T \in \mathcal{B}(X)$. If there exists a finite dimensional subspace $M \subseteq X$ such that M and $M + \operatorname{span}\{x\}$ are T-invariant for every $x \in X$, then $T = \alpha + F$ for some $\alpha \in \mathbb{C}$ and some finite rank operator F.*

Proof. Consider the operator $\widetilde{T} \colon X/M \to X/M$ defined by $\widetilde{T}(x+M) = Tx+M$. Since M is T-invariant, it follows that $\widetilde{T}$ is well-defined. Furthermore, since $M + \operatorname{span}\{x\}$ is also T-invariant for each $x \in X$, we see that each 1-dimensional subspace of X/M is $\widetilde{T}$-invariant. It follows that every subspace of X/M is also $\widetilde{T}$-invariant and hence $\widetilde{T}$ is scalar. Say $\widetilde{T} = \alpha$.

Define $F \in \mathcal{B}(X)$ by $Fx = Tx - \alpha x$. Then $FX \subseteq M$, so F is finite rank. □

Lemma 3.5. *Let X be a Banach space and $T \in \mathcal{B}(X)$. Suppose that every subspace is T-almost invariant. Then for each $x \in X$, the subspace $[T^n x]_{n\geqslant 0}$ is finite dimensional.*

Proof. Suppose not, and say that for some $x_1 \in X$, the subspace $[T^n x_1]$ is infinite dimensional. It follows that for each $k \geqslant 1$, $T^k x_1 \notin [T^n x_1]_{n<k}$. We wish to construct a subspace of X that is not T-almost invariant.

Pick $x_1^* \in X^*$ such that $x_1^*(x_1) \neq 0$ and define P_1 by $P_1 x = x - \frac{x_1^*(x)}{x_1^*(x_1)} x_1$. P_1 is a projection with $\ker P_1 = \operatorname{span}\{x_1\}$ and $\operatorname{im} P_1 = \ker x_1^*$. Define $x_2 = P_1 T x_1$. We have $\operatorname{span}\{x_1, Tx_1\} = \operatorname{span}\{x_1, x_2\}$ and $x_2 \notin \operatorname{span}\{x_1\}$ since $Tx_1 \notin \operatorname{span}\{x_1\}$.

We continue inductively to get sequences $\{x_n\}_n$ of vectors, $\{x_n^*\}$ of functionals, and $\{P_n\}_n$ of projections which satisfy:

i) $x_i^*(x_j) = 0$ if and only if $i \neq j$;

ii) $P_n(x) = x - \sum_{k=1}^n \frac{x_k^*(x)}{x_k^*(x_k)} x_k$ is the projection with $\operatorname{im} P_n = \bigcap_{k=1}^n \ker x_k^*$ and $\ker P_n = \operatorname{span}\{x_1, \cdots, x_n\}$;

iii) $x_{n+1} = P_n T x_n$;

iv) $\operatorname{span}\{x_1, \cdots, T^{n-1} x_1\} = \operatorname{span}\{x_1, \cdots, x_n\}$;

v) $x_n \notin \operatorname{span}\{x_1, \cdots, x_{n-1}\}$.

To show that we can indeed continue the construction, suppose we have x_k, x_{k-1}^*, and P_{k-1} for $k \leqslant n$ satisfying i) to v). Since $x_n \notin \operatorname{span}\{x_1, \cdots, x_{n-1}\}$, by the Hahn-Banach Theorem we can find $x_n^* \in X^*$ such that $x_n^*(x_n) \neq 0$ and $x_n^*(x_k) = 0$ for $k < n$. With this x_n^*, we may use ii) to define P_n.

Define $x_{n+1} = P_n T x_n$ to satisfy iii). Since $\operatorname{im} P_n = \bigcap_{k=1}^n \ker x_k^*$, we see that i) holds. By definition of P_n and x_{n+1}, there exists some $y_n \in \operatorname{span}\{x_1, \cdots, x_n\}$ such that $x_{n+1} = Tx_n + y_n$ and iv) shows that $x_n, y_n \in \operatorname{span}\{x_1, \cdots, T^{n-1} x_1\}$, hence $x_{n+1} \in \operatorname{span}\{x_1, \cdots, T^n x_1\}$. Also $T^{n-1} x_1 \in \operatorname{span}\{x_1, \cdots, x_n\}$ and $Tx_k \in \operatorname{span}\{x_1, \cdots, x_{k+1}\}$ for each $k \leqslant n$. Putting this together gives

$$T^n x_1 \in \operatorname{span}\{Tx_1, \cdots, Tx_n\} \subseteq \operatorname{span}\{x_1, \cdots, x_{n+1}\}. \tag{33}$$

Using the above and iv), we have $\operatorname{span}\{x_1, \cdots, T^n x_1\} = \operatorname{span}\{x_1, \cdots, x_{n+1}\}$. This shows that iv) holds.

The conditions show that $T^n x_1 \notin \operatorname{span}\{x_1, \cdots, T^{n-1} x_1\} = \operatorname{span}\{x_1, \cdots, x_n\}$ but $T^n x_1 \in \operatorname{span}\{x_1, \cdots, x_{n+1}\}$. Hence it follows that $x_{n+1} \notin \operatorname{span}\{x_1, \cdots, x_n\}$, so v) holds as well.

Set $Y = [x_{2n-1}]_{n \geqslant 0}$. By assumption, Y is T-almost invariant, so there exists some finite dimensional subspsace F such that $TY \subseteq Y + F$. As such, $Tx_{2n-1} = y_n + f_n$ for some $y_n \in Y$ and some $f_n \in F$. From earlier definitions, $P_{2n-1} T x_{2n-1} = x_{2n}$, so $Tx_{2n-1} = x_{2n} + u_n$ for some $u_n \in \operatorname{span}\{x_1, \cdots, x_{2n-1}\}$.

Pick $j > n$. By i), $x_{2j}^*(x_{2n}) = x_{2j}^*(u_n) = x_{2j}^*(y_n) = 0$ since Y is spanned by the odd indices $2k-1$ and $u_n \in \operatorname{span}\{x_1, \cdots, x_{2n-1}\}$. Since $y_n + f_n = Tx_{2n-1} = x_{2n} + u_n$, we have $x_{2j}^*(f_n) = 0$ as well. However, $x_{2n}^*(x_{2n}) \neq 0$ but $x_{2n}^*(u_n) = x_{2n}^*(y_n) = 0$. Similar reasoning shows that $x_{2n}^*(f_n) \neq 0$.

The x_k^* are linearly independent and we have $x_{2n}^*(f_n) \neq 0$ while $x_{2j}^*(f_n) = 0$ for each n and $j > n$. We conclude that F is infinite dimensional, contradicting our original assumptions. $\square$

With Lemmas 3.4 and 3.5, we are now able to prove Proposition 3.2.

Proof of Proposition 3.2. Suppose otherwise, then by Lemma 3.4, given any T-invariant finite dimensional subspace $M \subseteq X$, there exists some vector $x \in X$ such that $M + \operatorname{span}\{x\}$ is not T-invariant.

Start with the subspace $\{0\}$. There exists some vector $x_1 \in X$ such that $Tx_1 \notin \operatorname{span}\{x_1\}$. Set $M_1 = \operatorname{span}\{x_1\}$ and choose $x_1^* \in X^*$ such that $x_1^*|_{M_1} = 0$ and $x_1^*(Tx_1) \neq 0$. Set $M_1' = [T^n x_1]_{n \geqslant 0}$.

Lemma 3.5 says that M_1' is finite dimensional. Since M_1' is also T-invariant, there again exists some $x_2 \in X$ such that $M_1' + \operatorname{span}\{x_2\}$ is not T-invariant. Since $X = \ker x_1^* \oplus \operatorname{span}\{Tx_1\}$ and $Tx_1 \in M_1'$, we can choose x_2 such that $x_2 \in \ker x_1^*$.

We continue this inductively and construct sequences $\{x_n\}_n$ of vectors, $\{x_n^*\}$ of functionals, and $\{M_n\}$ and $\left\{M_n'\right\}$ of finite dimensional subspaces of X such that

i) $x_i^*(x_j) = 0$ for all i, j;

ii) $x_i^*(Tx_j) \neq 0$ if $i = j$ and $x_i^*(Tx_j) = 0$ if $i > j$;

iii) $M_n = M_{n-1}' + \operatorname{span}\{x_n\}$;

iv) $M_n' = M_n + [T^k x_n]_{k \geqslant 0}$ and M_n' is T-invariant.

To show that we can indeed continue this construction, suppose that we have defined x_k, x_k^*, M_k, and M_k' for $k \leqslant n$ which satisfy i) to iv). Since M_n' is finite dimensional and T-invariant, there exists some $x_{n+1} \in X$ such that $M_n' + \operatorname{span}\{x_{n+1}\}$ is not T-invariant.

ii) gives that

$$X = \bigcap_{k=1}^{n} \ker x_k^* \oplus \operatorname{span}\{Tx_1, \cdots, Tx_n\}. \tag{34}$$

iii) and iv) show that $\operatorname{span}\{Tx_1, \cdots, Tx_n\} \subseteq M_n'$. Thus we can pick x_{n+1} to be i $\bigcap_{k=1}^{n} \ker x_k^*$. Since M_n' is T-invariant, we must have that $Tx_{n+1} \notin M_n' + \operatorname{span}\{x_{n+1}\}$.

Let $M_{n+1} = M_n' + \operatorname{span}\{x_{n+1}\}$ to satisfy iii). Pick $x_{n+1}^* \in X^*$ such that $x_{n+1}^*|_{M_{n+1}} = 0$ and $x_{n+1}^*(Tx_{n+1}) \neq 0$. We will have that $x_{n+1}^*(x_k) = 0$ for each $k \leqslant n+1$ and $x_{n+1}^*(Tx_k) = 0$ for each $k \leqslant n$. This construction satisfies i) and ii) as well.

We define $M_{n+1}' = M_{n+1} + [T^k x_{n+1}]_{k \geqslant 0}$. Lemma 3.5 shows that M_{n+1}' is finite dimensional and it is clear that it is T-invariant. Condition iv) is now satisfied.

Define $Y = [x_n]_{n \geqslant 1}$. Y is T-almost invariant by assumption, so there exists some finite dimensional subspace F such that $TY \subseteq Y + F$. For each $x_n \in Y$, there exists some $y_n \in Y$ and some $f_n \in F$ such that $Tx_n = y_n + f_n$.

By the conditions above, we have $x_n^*(Tx_n) \neq 0$ and $x_n^*(y_n) = 0$. It follows that $x_n^*(f_n) \neq 0$. For $k > n$, $x_k^*(Tx_n) = x_k^*(y_n) = 0$, and so $x_k^*(f_n) = 0$ as well. Since the x_k^* are linearly independent, we must have that F is infinite dimensional - a contradiction. $\square$

Remark 3.6. Lemma 3.5 states that if every subspace is T-almost invariant for $T \in \mathcal{B}(X)$, then T is locally algebraic. An adaptation of Kaplansky's Lemma shows that if this is the case, then T is in fact algebraic. Using this idea, if an alternative proof to Lemma 3.5 were to be found, one could possibly prove Proposition 3.2 by circumventing the constructions from the above proofs.

The Reflexive Banach Space Case

The original results of Androulakis et al. have since been extended to show that every operator acting on a Banach space has an almost invariant half-space. This section works towards a proof of this result by looking at papers of Popov [2], of Tcaciuc [6], and of Popov and Tcaciuc [4].

The desired result was achieved in steps and the existence of almost invariant half-spaces was first proven for operators acting on a reflexive Banach space. The key idea in this proof uses basic sequences and the Kadets-Pełczyński Criterion, stated below without proof.

Theorem 4.1 (Kadets-Pełczyński Criterion)**.** *Let S be a bounded subset of a Banach space X such that 0 is not in the norm-closure of S. Then the following are equivalent:*

a) S fails to contain a basic sequence;

b) The WOT-closure of S is WOT-compact and does not contain 0.

Theorem 4.2. *Let X be a Banach space and $T \in \mathcal{B}(X)$. Suppose that there exists $\mu \in \partial\sigma(T)$ that is not an eigenvalue. Then T has an almost invariant half-space with defect at most 1.*

Proof. Without loss of generality, we may assume that $\mu = 0$ by replacing T with $T - \mu$ since this does not change the almost invariant subspaces of the operator or their defects. As in the proof of Theorems 2.6 and 2.10, we still construct vectors of the form $h(\lambda_n, e)$ but this time, we wish to extract a basic sequence from it, resulting in the desired almost invariant half-space.

Since $0 \in \partial\sigma(T)$, we may pick a sequence $\{\lambda_n\}_n$ in $\rho(T)$ converging to 0. We get a corresponding sequence $\{\lambda_n - T\}_n$ of invertible operators converging to a non-invertible one. This implies that the sequence of norms $\{\|(\lambda_n - T)^{-1}\|\}$ is unbounded. The Uniform Boundedness Principle gives the existence of some vector $e \in X$ such that

$$\|h(\lambda_n, e)\| = \|(\lambda_n - T)^{-1}e\| \to \infty. \tag{35}$$

By scaling, we may assume that $e \in \operatorname{ball} X$.

For simplicity, denote $h(\lambda_n, e)$ by h_n, $\frac{h_n}{\|h_n\|}$ by x_n and let $S = \{x_n\}_n \subseteq \operatorname{ball} X$. As before, we have the following

$$Tx_n = \frac{1}{\|h_n\|}Th_n = \lambda_n \frac{h_n}{\|h_n\|} - \frac{1}{\|h_n\|}e = \lambda_n x_n - \frac{1}{\|h_n\|}e. \tag{36}$$

Equation (36) shows that $[x_n]_{n\geqslant 0}$ is almost invariant with defect at most 1.

We now appealto the Kadets-Pełczyński Criterion (Theorem 4.1) by considering $\overline{S}^{\mathrm{WOT}}$.

- Case 1: $\overline{S}^{\mathrm{WOT}}$ is not WOT-compact.

  In this case it follows from the Kadets-Pełczyński Criterion that S contains a basic sequence. By passing to a subsequence, we may assume that $S = \{x_n\}_n$ is basic. By taking the subspace $Y = [x_{2n}]_{n\geqslant 0}$, we get an almost invariant subspace.

- Case 2: $\overline{S}^{\mathrm{WOT}}$ is WOT-compact.

  In this case, we still wish to extract a basic sequence from S. We do this by showing that $\overline{S}^{\mathrm{WOT}}$ contains 0. By the Eberlein-Šmulian Theorem, WOT-compactness is equivalent to WOT-sequential compactness. By passing to a subsequence, we may assume that $x_n \to^{\mathrm{WOT}} z$ for some $z \in \overline{S}^{\mathrm{WOT}}$, and so $Tx_n \to^{\mathrm{WOT}} Tz$. Since $\lambda_n \to 0$ and $\|h_n\| \to \infty$,

  $$Tx_n = \lambda_n x_n - \frac{1}{\|h_n\|}e \to 0. \tag{37}$$

  Hence $Tz = 0$. Since 0 is not an eigenvalue of T, $z = 0$ and so $x_n \to^{\mathrm{WOT}} 0$. The Kadets-Pełczyński Criterion again shows the existence of a basic sequence in S and we may continue as in Case 1.

$\square$

A corollary of Theorem 4.2 is the following:

Corollary 4.3. *Let X be a reflexive Banach space and $T \in \mathcal{B}(X)$. Suppose that either $\partial\sigma(T)\backslash\sigma_p(T)$ or $\partial\sigma(T^*)\backslash\sigma_p(T^*)$ is non-empty, then T has an almost invariant half-space.*

Proof. Since X is reflexive, $T^{**} = T$. The hypotheses show that either T or T^* has an almost invariant half-space by Theorem 4.2. The result follows from Proposition 1.7. $\square$

What remains is to remove the eigenvalue condition from the hypothesis.

Theorem 4.4. *Let X be a reflexive Banach space and $T \in \mathcal{B}(X)$. Then T has an almost invariant half-space of defect at most 1.*

Proof. In light of Corollary 4.3 we may assume that any point in $\partial\sigma(T) = \partial\sigma(T^*)$ is an eigenvalue for both T and T^*.

- Case 1: $\partial\sigma(T)$ has infinite cardinality.

  In this case, pick disjoint, countably infinite sequences $\{\lambda_n\}_n$ and $\{\mu_n\}_n$ of $\partial\sigma(T)$. By assumption, each of the λ_n and μ_n are eigenvalues for T and T^*, hence we can choose eigenvectors $\{x_n\}_n$ in X and $\{f_n\}$ in X^* such that $Tx_n = \lambda_n x_n$ and $T^* f_n = \mu_n f_n$.

  The sets $\{x_n\}_n$ and $\{f_n\}_n$ are linearly independent, so $Y = [x_n]_{n\geqslant 0}$ is infinite dimensional. Since Y is the closed linear span of eigenvectors of T, Y is T-invariant. We show that Y is infinite codimensional in X.

  For $n, k \geqslant 0$, we have

  $$\lambda_k f_n(x_k) = f_n(\lambda_k x_k) = f_n(Tx_k) = T^* f_n(x_k) = \mu_n f_n(x_k). \tag{38}$$

  Since we chose the sequences $\{\lambda_n\}_n$ and $\{\mu_n\}_n$ to be disjoint, we must have $f_n(x_k) = 0$ for all $n, k \geqslant 0$. So the linearly independent functionals $\{f_n\}_n$ annihilate Y, showing infinite codimensionality.

- Case 2: $\partial\sigma(T)$ is finite.

  Finiteness of $\partial\sigma(T)$ implies finiteness of $\sigma(T)$ and so $\partial\sigma(T) = \sigma(T)$. In this case, criterion b) from Theorem 2.6 holds.

  For any T-invariant subspace Y of X, we have that $\partial\sigma(T|_Y) \subseteq \sigma_a(T|_Y) \subseteq \sigma_a(T) \subseteq \sigma(T)$. Reasoning from the above paragraph shows that $\sigma(T|_Y)$ is also finite and $\sigma(T|_Y) \subseteq \sigma(T)$.

  For $n \in \mathbb{N}$, set $Y_n = \overline{T^n X}$, and set $Y_0 = X$. Each Y_n is invariant under T, $Y_{n+1} = \overline{TY_n}$, and $X \supseteq Y_1 \supseteq Y_2 \supseteq \cdots$. It is clear that for $j, n \geqslant 0$ and $y \in Y_j$ that $T^n y \in Y_{n+j}$.

  We may assume that each Y_n is infinite dimensional, otherwise let h be the smallest index for which Y_n is finite dimensional. Any half-space of Y_{h-1} containing Y_h would then be a T-almost invariant half-space.

  For any $n \geqslant 0$, we have $\sigma(T|_{Y_{n+1}}) \subseteq \sigma(T|_{Y_n})$. Finiteness of $\sigma(T)$ and non-emptyness of the spectrum show that there exists $k \geqslant 0$ such that

$\sigma(T|_{Y_n}) = \sigma(T|_{Y_k}) \neq \varnothing$ for each $n \geqslant k$. Since an almost invariant half-space under $T|_{Y_k}$ is an almost invariant half-space under T, we may assume without loss of generality that $k = 0$. By perturbing by a scalar, we may also assume that $0 \in \sigma(T)$.

We will show that either there exists a vector $z \in X$ with its orbit under T being a minimal sequence or that the restriction of T to some Y_j has dense range.

We may assume that Y_1 is finite codimensional in X, otherwise it would be an invariant half-space for T. It follows that $X = Y_1 \oplus Z$ for some finite dimensional subspace Z. If $Z = \{0\}$, then T has dense range. Otherwise, let $\{z_1, \cdots, z_k\}$ be a basis for Z and assume that the orbits of none of the z_j under T are minimal sequences.

For $1 \leqslant j \leqslant k$, let p_j be the smallest index such that $T^{p_j} z_j \in [T^n z_j]_{n \neq p_j}$. By an equivalent criterion for minimal sequences, $T^{p_j} z_j \in [T^n z_j]_{n > p_j}$, so $T^{p_j} z_j \in Y_{p_j+1}$. By setting $p_0 = \max\{p_1, \cdots, p_k\}$, it follows that $T^{p_0} z_j = T^{p_0 - p_j}(T^{p_j} z_j) \in T_{p_0+1}$ for each j. Since $\{z_1, \cdots, z_k\}$ is a basis for Z, $T^{p_0} Z \subseteq Y_{p_0+1}$.

For any $y \in Y_1$, we see that $T^{p_0} y \in Y_{p_0+1}$. Since $X = Y_1 \oplus Z$, $T^{p_0} X \subseteq T_{p_0+1}$. Taking closures gives $Y_{p_0} \subseteq Y_{p_0+1}$, so $Y_{p_0} = Y_{p_0+1}$ and $T|_{Y_{p_0}}$ has dense range.

We get two more subcases:

- Case 2.1: There is a vector z with orbit under T being a minimal sequence.

  Here, we meet the hypotheses of Theorem 2.6 and so T has an almost invariant half-space.

- Case 2.2: There exists some Y_j such that $T|_{Y_j}$ has dense range.

  If $S = T|_{Y_j} \colon Y_j \to Y_j$ has dense range, then S^* is injective. Hence $0 \in \sigma(S) = \sigma(S^*) = \partial\sigma(S^*)$ and 0 is not an eigenvalue for S^*. Theorem 4.2 shows that S^* has an almost invariant half-space with defect at most 1. By reflexivity, so does S and T.

□

Remark 4.5. We note that some changes to the arguments presented in the proof of Theorem 4.4 are necessary if we consider a Hilbert space $\mathcal{H}$ instead. This is due to the differences in the definition of Banach space and Hilbert space adjoints.

The assumption made at the start of the proof changes to assuming that any point in $\partial\sigma(T)$ is an eigenvalue for T and any point in $\partial\sigma(T^*) = \overline{\partial\sigma(T)}$ is an eigenvalue for T^*. Also, instead of finding disjoint sequences in $\rho(T)$ for Case 1, we find disjoint sequences $\{\lambda_n\}_n$ in $\rho(T)$ and $\{\mu_n\}_n$ in $\rho(T^*) = \overline{\rho(T)}$. This is still possible, so Theorem 4.4 holds for Hilbert spaces as well.

The General Banach Space Case

We now want to remove the restriction of X being a reflexive Banach space. We initially required reflexivity so that information about T^* gave information about T. The missing link to this is given by a weak*-analog of the Bessaga-Pełczyński Selection Principle.

Theorem 5.1 (Bessaga-Pełczyński Selection Principle)**.** *Let $\{x_n^*\}_n$ be a semi-normalized ($0 < \inf_{n\geqslant 0}\|x_n^*\| \leqslant \sup_{n\geqslant 0}\|x_n^*\| < \infty$) weak*-null sequence in a dual Banach space X^*. Then there exists a basic subsequence $\{y_n^*\}_n$ of $\{x_n^*\}_n$ and a bounded sequence $\{y_n\}_n$ in X such that $y_i^*(y_j) = \delta_{ij}$ for all i, j.*

We also require another result prior to proving the general case.

Proposition 5.2. *Let X be a separable Banach space, then the unit ball of X^* is weak*-metrizable.*

Theorem 5.3. *Let X be a separable Banach space and $T \in \mathcal{B}(X)$. Suppose that $\partial\sigma(T^*)\backslash\sigma_p(T^*)$ is non-empty. Then T has an almost invariant half-space with defect at most 1.*

Proof. Let $\mu \in \partial\sigma(T^*)\backslash\sigma_p(T^*)$. By shifting by a scalar, we may assume without loss of generality that $\mu = 0$. Let $\{\lambda_n\}_n$ be a sequence in $\rho(T^*)$ converging to 0. Then $\|(\lambda_n - T^*)^{-1}\| \to \infty$, so by the Uniform Boundedness Principle, there exists a vector $e^* \in X^*$ such that $\|(\lambda_n - T^*)^{-1}e^*\| \to \infty$. Set $h_n^* = (\lambda_n - T^*)^{-1}e^*$ and $x_n^* = \frac{h_n^*}{\|h_n^*\|}$.

The sequence $\{x_n^*\}_n$ is normalized, hence semi-normalized, and

$$T^*x_n^* = \lambda_n x_n^* - \frac{1}{\|h_n^*\|}e^*. \tag{39}$$

By the Banach-Alaoglu Theorem, $\operatorname{ball} X^*$ is weak*-compact. Separability of X and Proposition 5.2 show that $\operatorname{ball} X^*$ is also weak*-metrizable. This gives weak*-sequential compactness of $\operatorname{ball} X^*$. Since $\{x_n^*\}_n$ consists of unit norm functionals, it is contained in $\operatorname{ball} X^*$, and by passing to a subsequence, we may assume that $x_n^* \to^{\text{weak}^*} y^*$ for some $y^* \in X^*$.

We have $\lambda_n \to 0$, $x_n^* \to^{\text{weak}^*} y^*$ and $\|h_n^*\| \to \infty$. Hence $T^*x_n^* \to^{\text{weak}^*} T^*y^*$ and Equation (39) show that $T^*x_n^* = \lambda_n x_n^* - \frac{1}{\|h_n^*\|}e^* \to 0$. This gives $T^*y^* = 0$, and since 0 is not an eigenvalue for T^*, y^* must be 0. Thus $\{x_n^*\}_n$ is weak*-null.

Since $\{x_n^*\}_n$ is weak*-null, we may apply the Bessaga-Pełczyński Selection Principle (Theorem 5.1). Passing once again to a subsequence, we may also assume that $\{x_n^*\}_n$ is a basic sequence and there exists a corresponding sequence $\{x_n\}_n$ in X such that $x_n^*(x_k) = \delta_{nk}$ for all n, k. Since $\{x_n^*\}_n$ is basic, it is linearly independent. The biorthogonality condition from the Selection Principle shows that $\{x_n\}_n$ is also linearly independent.

$[x_{2n+1}]_{n\geqslant 0} \subseteq ([x_{2n}^*]_{n\geqslant 0})_\perp$, so $([x_{2n}^*]_{n\geqslant 0})_\perp$ is infinite dimensional. Furthermore, each x_{2k}^* annihilates $([x_{2n}^*]_{n\geqslant 0})_\perp$, giving infinite codimensionality. Hence $([x_{2n}^*]_{n\geqslant 0})_\perp$ is a half-space.

From above, we see that by passing to another subsequence, we may assume that $\{x_n^*\}_n$ is a basic sequence and $Y = ([x_n^*]_{n\geqslant 0})_\perp = ([h_n^*]_{n\geqslant 0})_\perp$ is a half-space of X.

If $y \in Y$, then for each $n \geqslant 0$,

$$h_n^*(Ty) = T^*h_n^*(y) = (\lambda_n h_n^* - e^*)(y) = -e^*(y). \tag{40}$$

So, if $Y \subseteq \ker e^*$, then $h_n^*(Ty) = 0$ for each $n \geqslant 0$. In this case, $TY \subseteq Y$ so Y is T-invariant.

Otherwise, there exists some $y_0 \in Y$ with $y_0 \notin \ker e^*$. Set $f = Ty_0$ and for $y \in Y$, define a scalar $\alpha_y = \frac{e^*(y)}{e^*(y_0)}$. Then, for each $n \geqslant 0$,

$$\begin{aligned} h_n^*(Ty - \alpha_y f) &= h_n^*(Ty - \frac{e^*(y)}{e^*(y_0)} f) \\ &= h_n^*(Ty) - \frac{e^*(y)}{e^*(y_0)} h_n^*(Ty_0) \\ &= -e^*(y) + \frac{e^*(y)}{e^*(y_0)} e^*(y_0) \\ &= 0. \end{aligned} \tag{41}$$

Hence for each $y \in Y$, $Ty - \alpha_y f \in Y$. Hence $Ty \in Y + \mathrm{span}\{f\}$ and Y is T-almost invariant with defect at most 1. □

Combining the proofs of Theorems 4.2 and 4.4 and Corollary 4.3 with the above result establishes the following result:

Theorem 5.4. *Let X be a separable Banach space and $T \in \mathcal{B}(X)$. Then T has an almost invariant half-space with defect at most 1.*

Algebras and Lattices

We now discuss lattices of subspaces of a Hilbert space $\mathcal{H}$ and subalgebras of $\mathcal{B}(\mathcal{H})$. In particular, we look at properties of invariance and almost invariance held by these algebras and lattices. Though we restrtict our attention to Hilbert spaces, some of these results will apply in the general Banach space case.

Definition 6.1. Let $\mathcal{H}$ be a Hilbert space. For two subspaces M and N of $\mathcal{H}$, we define the **join** and **meet of** M **and** N respectively by

$$\begin{aligned} M \vee N &= \overline{M + N} \\ M \wedge N &= M \cap N \end{aligned} \tag{42}$$

Definition 6.2. Let $\mathcal{H}$ be a Hilbert space. If $\mathcal{A}$ is a subset of $\mathcal{B}(\mathcal{H})$, we define the **lattice of invariant subspaces of** $\mathcal{A}$ by

$$\operatorname{Lat} \mathcal{A} = \{M \subseteq \mathcal{H} \colon M \text{ is a } T\text{-invariant subspace for all } T \in \mathcal{A}\}\,. \tag{43}$$

If $\mathcal{L}$ is a collection of subspaces of $\mathcal{H}$, we define the **algebra of invariant operators for** $\operatorname{Lat} \mathcal{L}$ by

$$\operatorname{Alg} \mathcal{L} = \left\{T \in \mathcal{B}(\mathcal{H}) \colon M \text{ is a } T\text{-invariant subspace for all } M \in \mathcal{L}\right\}. \tag{44}$$

We extend this notion to almost invariant subspaces as follows:

$$\operatorname{Lat}_a \mathcal{A} = \{M \subseteq \mathcal{H} \colon M \text{ is a } T\text{-almost invariant subspace for all } T \in \mathcal{A}\} \tag{45}$$

$$\operatorname{Alg}_a \mathcal{L} = \left\{T \in \mathcal{B}(H) \colon M \text{ is a } T\text{-almost invariant subspace for all } M \in \mathcal{L}\right\}. \tag{46}$$

We note some basic properties of Alg and Lat and check if they extend to their almost invariant counterparts.

Proposition 6.3. *Let $\mathcal{H}$ be a Hilbert space, $\mathcal{A}$ be a subset of $\mathcal{B}(\mathcal{H})$, and $\mathcal{L}$ be a collection of subspaces of $\mathcal{H}$. Then*

a) $\operatorname{Lat} \mathcal{A}$ *is a complete lattice under the operations* $\vee$ *and* $\wedge$*;*

b) $\operatorname{Alg} \mathcal{L}$ *is a WOT-closed subalgebra of* $\mathcal{B}(\mathcal{H})$*;*

c) $\mathcal{L} \subseteq \operatorname{Lat} \operatorname{Alg} \mathcal{L}$ *and* $\mathcal{A} \subseteq \operatorname{Alg} \operatorname{Lat} \mathcal{A}$*;*

d) $\operatorname{Alg} \operatorname{Lat} \operatorname{Alg} \mathcal{L} = \operatorname{Alg} \mathcal{L}$ *and* $\operatorname{Lat} \operatorname{Alg} \operatorname{Lat} \mathcal{A} = \operatorname{Lat} \mathcal{A}$.

We will work towards an almost invariant analog of Proposition 6.3 and show that properties c) and d) in fact still hold in that case. Unfortunately, in the almost invariant setting, we lose completeness of $\operatorname{Lat} \mathcal{A}$ in property a) and lose WOT-closure of $\operatorname{Alg} \mathcal{L}$ in property b). In order to show the above, we first require an alternative way to view almost invariant subspaces, shown by Assadi, Farzaneh, and Mohammadinejad [7].

Proposition 6.4. *Let X be a Banach space, $Y \subseteq X$ be a subspace and $T \in \mathcal{B}(X)$. Y is T-almost invariant if and only if there exists a finite codimensional subspace $N \subseteq X$ such that $T(Y \cap N) \subseteq Y$. Moreover, if N is a subspace of minimum codimension, then* $\operatorname{codim} N$ *is equal to the defect of Y.*

Proof. Let Y be T-almost invariant and M be a minimal error space. As in previous proofs (Proposition 1.4), we note the sum $Y + M$ is direct.

Let $q\colon X \to X/Y$ be the quotient map. Then $q(M)$ is a finite dimensional subspace of X/Y, and so there exists a subspace $L' \subseteq X/Y$ such that $X/Y = L' \oplus q(M)$. Since $Y \cap M = \{0\}$, taking the preimage $L = q^{-1}(L')$, we get $X = L \oplus M$ and $Y \subseteq L$.

$T^{-1}L$ is a subspace of X since T is continuous and L is a subspace. Define $\widetilde{T}\colon X/T^{-1}L \to X/L$ by $\widetilde{T}(x + T^{-1}L) = Tx + L$. This map is well defined and injective, hence we get

$$\operatorname{codim} T^{-1}L = \dim X/T^{-1}L \leqslant \dim X/L = \dim M < \infty. \tag{47}$$

Set $N = T^{-1}L$. By Equation 47, N is finite codimensional. We also see that

$$T(Y \cap N) = T(Y \cap T^{-1}L) \subseteq TY \cap L \subseteq (Y + M) \cap L = Y. \tag{48}$$

For the converse, suppose that N is a finite comdiensional subspace of X such that $T(Y \cap N) \subseteq Y$. We can pick N to be of minimal codimension.

$X = Y + N$, since otherwise, there would exists some $x \in X \backslash (Y + N)$. If this were the case, we can take $N' = N + \operatorname{span}\{x\}$ so $Y \cap N' = Y \cap N$ and $T(Y \cap N') \subseteq Y$ but $\operatorname{codim} N' < \operatorname{codim} N$, contradicting minimality.

By finite codimensionality of N, we can pick some finite dimensional subspace M_1 of X such that $X = M_1 \oplus N$ and $M_1 \subseteq Y$. Here,

$$TY = T((Y \cap N) + M_1) \subseteq T(Y \cap N) + TM_1 \subseteq Y + TM_1 \tag{49}$$

hence Y is T-almost invariant.

For the final assertion, let N be a subspace of minimal codimension such that $N \supseteq T^{-1}L$ and k be the defect of Y under T. We have

$$k \leqslant \dim TM_1 \leqslant \dim M_1 = \operatorname{codim} N \leqslant \operatorname{codim} T^{-1}L \leqslant \operatorname{codim} L = \dim M = k. \tag{50}$$

□

With this alternative definition of almost invariance, we are able to prove the almost invariant analog of Proposition 6.3.

Proposition 6.5. *Let $\mathcal{H}$ be a Hilbert space, $\mathcal{A}$ be a subset of $\mathcal{B}(\mathcal{H})$, and $\mathcal{L}$ be a collection of subspaces of $\mathcal{H}$. Then*

a) $\operatorname{Lat}_a \mathcal{A}$ *is a lattice under the operations* $\vee$ *and* $\wedge$*;*

b) $\operatorname{Alg}_a \mathcal{L}$ *is a subalgebra of* $\mathcal{B}(\mathcal{H})$*;*

c) $\mathcal{L} \subseteq \operatorname{Lat}_a \operatorname{Alg}_a \mathcal{L}$ *and* $\mathcal{A} \subseteq \operatorname{Alg}_a \operatorname{Lat}_a \mathcal{A}$*;*

d) $\operatorname{Alg}_a \operatorname{Lat}_a \operatorname{Alg}_a \mathcal{L} = \operatorname{Alg}_a \mathcal{L}$ *and* $\operatorname{Lat}_a \operatorname{Alg}_a \operatorname{Lat}_a \mathcal{A} = \operatorname{Lat}_a \mathcal{A}$*.*

Proof. a) It suffices to show that for some $T \in \mathcal{B}(\mathcal{H})$ that the join and meet of two T-almost invariant are also T-almost invariant.

Suppose $TY_1 \subseteq Y_1 + M_1$ and $TY_2 \subseteq Y_2 + M_2$ for finite dimensional subspaces M_1 and M_2 of $\mathcal{H}$. Then, $T(Y_1 + Y_2) \subseteq Y_1 + Y_2 + M_1 + M_2$. Since M_1 and M_2 are finite dimensional, so is $M_1 + M_2$, and so

$$T(\overline{Y_1 + Y_2}) \subseteq \overline{Y_1 + Y_2} + M_1 + M_2.$$

This shows $Y_1 \vee Y_2$ is T-almost invariant.

By Proposition 6.4, there exist finite codimensional subspaces N_1 and N_2 of $\mathcal{H}$ such that $T(Y_1 \cap N_1) \subseteq Y_1$ and $T(Y_2 \cap N_2) \subseteq Y_2$. Hence

$$T(Y_1 \cap Y_2 \cap N_1 \cap N_2) \subseteq T(Y_1 \cap N_1) \cap T(Y_2 \cap N_2) \subseteq Y_1 \cap Y_2.$$

This shows $Y_1 \wedge Y_2$ is T-almost invariant.

b) It suffices to consider the case where $\mathcal{L}$ is a singleton. Say $\mathcal{L} = \{Y\}$ and write $\operatorname{Alg}_a Y$ instead of $\operatorname{Alg}_a \mathcal{L}$.

Suppose that $T_1, T_2 \in \operatorname{Alg}_a Y$, then $T_1 Y \subseteq Y + M_1$ and $T_2 Y \subseteq Y + M_2$ for finite dimensional subspaces M_1 and M_2 of $\mathcal{H}$. Then for $\alpha \in \mathbb{C}$,

$$(T_1 + \alpha T_2)Y \subseteq Y + M_1 + M_2$$

and

$$T_1 T_2 Y \subseteq T_1(Y + M_2) \subseteq Y + M_1 + T_1 M_2.$$

Since both $M_1 + M_2$ and $M_1 + T_1 M_2$ are finite dimensional, Y is both $(T_1 + \alpha T_2)$-almost invariant and $(T_1 T_2)$-almost invariant. Hence $\operatorname{Alg}_a Y$ is a subalgebra of $\mathcal{B}(\mathcal{H})$.

c) Clearly every subspace $M \in \mathcal{L}$ is T-almost invariant for each $T \in \operatorname{Alg}_a \mathcal{L}$. This shows the first containment. The second follows similarly.

d) By c), we get $\operatorname{Alg}_a \operatorname{Lat}_a \operatorname{Alg}_a \mathcal{L} \subseteq \operatorname{Alg}_a \mathcal{L}$. We get the other inclusion by applying Part c) to the algebra $\operatorname{Alg}_a \mathcal{L}$. The other equality follows similarly.

□

To show why Proposition 6.5 cannot be a direct mirror of Proposition 6.3, we provide counterexamples to the completeness of $\mathrm{Lat}_a \mathcal{A}$ and the WOT-closure of $\mathrm{Alg}_a \mathcal{L}$.

Example 6.6. As a counterexample to the completeness of $\mathrm{Lat}_a \mathcal{A}$, let $\mathcal{H}_1$ and $\mathcal{H}_2$ be separable infinite dimensional Hilbert spaces. Let $\mathcal{A}$ be the singleton containing the operator $T \in \mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ of the form

$$T = \begin{bmatrix} 0 & 0 \\ I & 0 \end{bmatrix}.$$

Fix an orthonormal basis $\{e_0, e_1, e_2, \cdots\}$ for $\mathcal{H}_1$ and consider the chain of almost invariant subspaces $\left\{[e_k]_{k \leqslant n}\right\}_n$. This chain is not complete since $\mathcal{H}_1$ is not T-almost invariant.

Example 6.7. To show that $\mathrm{Alg}_a \mathcal{L}$ is not WOT-closed in general we use the scalar-plus-finite-rank characterization (Proposition 3.2).

Let $\mathcal{L}$ be the lattice of all subspaces of an infinite dimensional separable Hilbert space $\mathcal{H}$. The scalar-plus-finite-rank characterization shows that $\mathrm{Alg}_a \mathcal{L}$ are exactly the scalar-plus-finite-rank operators in $\mathcal{B}(\mathcal{H})$.

Consider a Donoghue operator $D \in \mathcal{B}(\ell_2(\mathbb{N}))$. D is compact, and hence is the norm-limit of a sequence of finite rank operators. Being a weighted shift with non-zero weights, the half-space $[x_{2n}]_{n \geqslant 0}$ is not D-almost invariant. This shows that $\mathrm{Alg}_a \mathcal{L}$ is not norm-closed, and hence not WOT-closed either.

Reflexivity

The counterexamples presented at the end of the previous section show that the lattice of almost invariant subspaces and algebra of almost invariant operators are weaker notions than their invariant counterparts. This is somewhat to be expected since we gain the freedom to perturb by finite dimensional subspaces and finite rank operators.

Despite losing completeness of the lattice and WOT-closure of the algebra, we continue to study this idea and extend it to reflexivity before looking at other properties of reflexivity. As in the previous section, we restrict our attention to Hilbert spaces, though some of the ideas that follow are applicable to general Banach spaces.

We start with some preliminary definitions and propositions reagarding the notion of reflexivity for subspaces and algebras.

Definition 7.1. Let $\mathcal{H}$ be a Hilbert space. For a subspace $\mathcal{S}$ of $\mathcal{B}(\mathcal{H})$, define

$$\operatorname{Ref}\mathcal{S} = \left\{T \in \mathcal{B}(\mathcal{H})\colon Th \in [\mathcal{S}h] \text{ for all } h \in \mathcal{H}\right\}. \tag{51}$$

We refer to $\operatorname{Ref} S$ as the **reflexive closure of** $\mathcal{S}$.

Proposition 7.2. *Let $\mathcal{H}$ be a Hilbert space and $\mathcal{S}$ be a subspace of $\mathcal{B}(\mathcal{H})$. Then*

a) $\operatorname{Ref}\mathcal{S}$ *is a SOT-closed subspace of* $\mathcal{B}(\mathcal{H})$*;*

b) $\operatorname{Ref}\operatorname{Ref}\mathcal{S} = \operatorname{Ref}\mathcal{S}$*;*

c) If $\mathcal{S}^* = \left\{T^* \in \mathcal{B}(\mathcal{H})\colon T \in \mathcal{S}\right\}$*, then* $\operatorname{Ref}\mathcal{S}^* = (\operatorname{Ref}\mathcal{S})^*$*.*

Proof. a) It is clear that $\operatorname{Ref}\mathcal{S}$ is a subspace of $\mathcal{B}(\mathcal{H})$, thus it is convex. We show that $\operatorname{Ref}\mathcal{S}$ is WOT-closed, hence SOT-closed.

Let $\{T_\alpha\}_\alpha$ be a net in $\operatorname{Ref}\mathcal{S}$ and suppose that $T_\alpha \to^{\text{WOT}} T$. Fix $h \in \mathcal{H}$. For each $g \in [\mathcal{S}h]^\perp$ we have $\langle T_\alpha h, g\rangle = 0$ for each α. Since $T_\alpha \to^{\text{WOT}} T$, it follows that $\langle Th, g\rangle = 0$ and hence $Th \in [\mathcal{S}h]$.

b) It is clear that $\mathcal{S} \subseteq \operatorname{Ref}\mathcal{S}$ and hence $\operatorname{Ref} S \subseteq \operatorname{Ref}\operatorname{Ref} S$. For the other inclusion, suppose that $T \in \operatorname{Ref}\operatorname{Ref} S$. By definition, for every $h \in \mathcal{H}$, $Th \in [\operatorname{Ref}\mathcal{S}h]$. Hence there is a sequence $\{A_n\}_n$ in $\operatorname{Ref}\mathcal{S}$ such that $A_n h \to Th$. Since each $A_n \in \operatorname{Ref}\mathcal{S}$, $A_n \in [\mathcal{S}h]$, and so $Th \in [\mathcal{S}h]$. Since this holds for arbitrary $h \in \mathcal{H}$, $T \in \operatorname{Ref} S$.

c) Suppose that $T \in \operatorname{Ref}\mathcal{S}$. Consider T^* and fix $h \in \mathcal{H}$. For any $g \in [\mathcal{S}^*h]^\perp$, we have for each $A^* \in \mathcal{S}^*$, $\langle A^*h, g\rangle = 0 = \langle h, Ag\rangle$. This shows $h \in [\mathcal{S}g]^\perp$. Since $T \in \operatorname{Ref}\mathcal{S}$, if follows that $\langle h, Tg\rangle = 0 = \langle T^*h, g\rangle$, and $T^*h \in [\mathcal{S}^*h]$. Since h was arbitrary, we get $T^* \in \operatorname{Ref}\mathcal{S}^*$, giving $(\operatorname{Ref}\mathcal{S})^* \subseteq \operatorname{Ref}\mathcal{S}^*$. We get the other inclusion since $(\operatorname{Ref}\mathcal{S}^*)^* \subseteq \operatorname{Ref}\mathcal{S}^{**} = \operatorname{Ref}\mathcal{S}$.

□

Proposition 7.3. *Let $\mathcal{H}$ be a Hilbert space. If $\mathcal{A}$ is a subalgebra of $\mathcal{B}(\mathcal{H})$ that contains the identity, then* $\operatorname{Alg}\operatorname{Lat}\mathcal{A} = \operatorname{Ref} A$.

Proof. Let $T \in \operatorname{Ref}\mathcal{A}$ and $M \in \operatorname{Lat}\mathcal{A}$. For any $h \in M$, $Th \in [\mathcal{A}h] \subseteq M$. So $T \in \operatorname{Alg}\operatorname{Lat}\mathcal{A}$.

For the other inclusion, assume that $T \in \operatorname{Alg}\operatorname{Lat}\mathcal{A}$ and $h \in \mathcal{H}$. Since $\mathcal{A}$ contains the identity, $h \in [\mathcal{A}h]$ and $[\mathcal{A}h] \in \operatorname{Lat}\mathcal{A}$. This shows that $Th \in [\mathcal{A}h]$ and hence $T \in \operatorname{Ref}\mathcal{A}$. □

With the definition of reflexive closure and the previous propositions, we are now able to define reflexivity of subspaces and subalgebras of $\mathcal{B}(\mathcal{H})$.

Definition 7.4. Let $\mathcal{H}$ be a Hilbert space. A subspace $\mathcal{S}$ of $\mathcal{B}(\mathcal{H})$ is **reflexive** if $\mathcal{S} = \operatorname{Ref}\mathcal{S}$. A single operator $T \in \mathcal{B}(\mathcal{H})$ is reflexive if the WOT-closed algebra generated by T and the identity is reflexive.

By Proposition 7.3, we see that a subalgebra $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$ containing the identity is reflexive if $\mathcal{A} = \operatorname{Alg}\operatorname{Lat}\mathcal{A}$. We define reflexivity of lattices of subspaces of $\mathcal{H}$ similarly: A lattice $\mathcal{L}$ of subspaces of $\mathcal{H}$ is reflexive if $\mathcal{L} = \operatorname{Lat}\operatorname{Alg}\mathcal{L}$.

We also extend the notion of reflexivity to the almost invariant setting. We shall say a subalgebra $\mathcal{A}$ of $\mathcal{B}(\mathcal{H})$ is **almost reflexive** if $\mathcal{A} = \operatorname{Alg}_a \operatorname{Lat}_a \mathcal{A}$. Similarly, a lattice $\mathcal{L}$ of subspaces of $\mathcal{H}$ is **almost reflexive** if $\mathcal{L} = \operatorname{Lat}_a \operatorname{Alg}_a \mathcal{L}$.

Proposition 7.5. *Let $\mathcal{H}$ and $\mathcal{H}_n$ be a Hilbert spaces for each n.*

a) If $\{\mathcal{S}_n\}_n$ is a sequence of reflexive subspaces with $\mathcal{S}_n \subseteq \mathcal{B}(\mathcal{H}_n)$ for each n, then $\oplus_n \mathcal{S}_n$ is reflexive;

b) If $\{\mathcal{S}_n\}_n$ is any collection of reflexive subspaces with $\mathcal{S}_n \subseteq \mathcal{B}(\mathcal{H})$ for each n, then $\bigcap_n \mathcal{S}_n$ is reflexive.

Proof. a) Note that if $\mathcal{S} = \oplus_n \mathcal{S}_n$ and $\mathcal{H} = \oplus_n \mathcal{H}_n$, then $[\mathcal{S}h_n] \subseteq \mathcal{H}_n$ for any $h_n \in \mathcal{H}_n$. So, if $T \in \operatorname{Ref}\mathcal{S}$, $T\mathcal{H}_n \subseteq \mathcal{H}_n$ for each n. This means T decomposes as a direct sum $T = \oplus_n T_n$. $T_n \in \operatorname{Ref}\mathcal{S}_n$ for each n, and since each $\mathcal{S}_n$ is reflexive, $T_n \in \mathcal{S}_n$. This shows that $T \in \mathcal{S}$.

b) If $\mathcal{S} = \bigcap_n \mathcal{S}_n$ and $T \in \operatorname{Ref}\mathcal{S}$, then $Th \in [\mathcal{S}h] \subseteq [\mathcal{S}_n h]$ for every vector $h \in \mathcal{H}$. Hence $T \in \operatorname{Ref}\mathcal{S}_n = \mathcal{S}_n$ for each n and so $T \in \mathcal{S}$.

□

In the almost reflexive setting (restricting our attention to algebras), we see that b) still holds while a) does not. We present a counterexample to a) first before proving b).

Example 7.6. Let $\mathcal{H}$ be an infinite dimensional separable Hilbert space. By the scalar-plus-finite-rank characterization (Proposition 3.2), given any collection $\mathcal{L}$ of subspaces of $\mathcal{H}$, $\operatorname{Alg}_a \mathcal{L}$ will contain all scalar-plus-finite-rank operators in $\mathcal{B}(\mathcal{H})$.

Consider $\mathcal{H} \oplus \mathcal{H}$ and choose any almost reflexive algebras $\mathcal{A}_1$ and $\mathcal{A}_2$ in $\mathcal{B}(\mathcal{H})$. The observation above notes that $\operatorname{Alg}_a \operatorname{Lat}_a \mathcal{A}_1 \oplus \mathcal{A}_2$ contains all scalar-plus-finite-rank operators in $\mathcal{B}(\mathcal{H} \oplus \mathcal{H})$.

Any operator $T \in \mathcal{A}_1 \oplus \mathcal{A}_2$ would have the form

$$T = \begin{bmatrix} T_{11} & 0 \\ 0 & T_{22} \end{bmatrix}$$

where $T_{11} \in \mathcal{A}_1$ and $T_{22} \in \mathcal{A}_2$. However, there exists finite rank operators in $\mathcal{B}(\mathcal{H} \oplus \mathcal{H})$ with the form

$$F = \begin{bmatrix} 0 & 0 \\ F_{12} & 0 \end{bmatrix}$$

where F_{12} is finite rank. Clearly $F \notin \mathcal{A}_1 \oplus \mathcal{A}_2$ hence $\mathcal{A}_1 \oplus \mathcal{A}_2$ is not almost reflexive.

The previous example does not bode well for the notion of almost reflexivity since even finite direct sums of almost reflexive algebras are in general not almost reflexive.

Proposition 7.7. *Let $\mathcal{H}$ be a Hilbert space. If $\{\mathcal{A}_n\}_n$ is any collection of almost reflexive algebras with $\mathcal{A}_n \subseteq \mathcal{B}(\mathcal{H})$ for each n, then $\bigcap_n \mathcal{A}_n$ is almost reflexive.*

Proof. By Part c) of Proposition 6.5, $\bigcap_n \mathcal{A}_n \subseteq \mathrm{Alg}_a \mathrm{Lat}_a \bigcap_n \mathcal{A}_n$. For the other containment, note that for each k, we have $\bigcap_n \mathcal{A}_n \subseteq \mathcal{A}_k$. Hence $\mathrm{Lat}_a \mathcal{A}_k \subseteq \mathrm{Lat}_a \bigcap_n \mathcal{A}_n$, and $\mathrm{Alg}_a \mathrm{Lat}_a \bigcap_n \mathcal{A}_n \subseteq \mathrm{Alg}_a \mathrm{Lat}_a \mathcal{A}_k = \mathcal{A}_k$, where the final equality comes from almost reflexivity. Since $\mathrm{Alg}_a \mathrm{Lat}_a \bigcap_n \mathcal{A}_n \subseteq \mathcal{A}_k$ for each k, the result follows. $\square$

Remark 7.8. It is at this point where I am unsure on whether this notion of almost reflexivity is useful. Nice properties that reflexive algebras hold are not shared by almost reflexive algebras and as previously stated, the lattice of almost invariant subspaces not being complete as well as the algebra of almost invariant operators not being WOT-closed will pose issues.

Altering the definition of almost reflexivity could make it a better idea to study, however it is unclear on what the best way to approach a new definition would be. One possible idea would be to restrict the size of the defects by defining something such as

$$\begin{aligned} \mathrm{Alg}_k \mathcal{L} = \{T \in \mathcal{B}(H) \colon & M \text{ is a } T\text{-almost invariant subspace} \\ & \text{with defect at most } k \text{ for all } M \in \mathcal{L}\} \end{aligned}$$

for some $k \geqslant 1$.

There are some issues with the above as it is not clear whether $\mathrm{Alg}_k \mathcal{L}$ is a subspace of $\mathcal{B}(H)$. Even if it were, we know the existence of operators T without invariant half-spaces that have almost invariant half-spaces with defect at most 1. If Y were a T-almost invariant half-space, then $TY \subseteq Y + F$ for some 1-dimensional subspace F. It follows that $T^2 Y \subseteq Y + F + TF$. TF cannot be contained in $Y + F$, otherwise $Y + F$ would be an invariant half-space.

Continuing this process k times, we eventually see that $T^{k+1} \notin \mathrm{Alg}_k\mathcal{L}$ and that $\mathrm{Alg}_k\mathcal{L}$ is in general not an algebra.

We now prove a few more properties of reflexivity.

Proposition 7.9. *Let $\mathcal{H}$ be a Hilbert space. Every 1-dimensional subspace of $\mathcal{B}(\mathcal{H})$ is reflexive.*

Proof. Let $\mathcal{S} = \mathrm{span}\{T\}$ for some non-zero $T \in \mathcal{B}(\mathcal{H})$. If $A \in \mathrm{Ref}\,\mathcal{S}$, then for every vector $h \in \mathcal{H}$, $Ah \in [\mathcal{S}h] = \mathrm{span}\{Th\}$. Hence there exists a scalar $\alpha(h)$ such that $Ah = \alpha(h)Th$.

For vectors $g, h \in \mathcal{H}$,

$$\begin{aligned} Ag &= \alpha(g)Tg \\ Ah &= \alpha(h)Th \\ A(g+h) &= \alpha(g+h)(Tg+Th). \end{aligned} \tag{52}$$

By comparing the Equations (52), we see that there exists a single scalar α such that $Ah = \alpha Th$ for each $h \in \mathcal{H}$, and so $A = \alpha T$.

Indeed, to see this, we rearrange and substitute Equations (52) into each other to get

$$\begin{aligned} \alpha(g)Tg = Ag = \alpha(g+h)Tg + [\alpha(g+h) - \alpha(h)]Th \\ \alpha(h)Th = Ah = \alpha(g+h)Th + [\alpha(g+h) - \alpha(g)]Tg. \end{aligned} \tag{53}$$

Adding Equations (53) together and comparing it with Equations (52) shows that

$$\alpha(g+h)(Tg+Th) - \alpha(g)Tg - \alpha(h)Th = 0. \tag{54}$$

Without loss of generality, we may assume that $Tg \neq 0 \neq Th$ since otherwise we may pick α to be any arbitrary scalar. If Tg and Th are linearly independent, then Equations (53) show that $\alpha(g) = \alpha(h) = \alpha(g+h)$.

Otherwise, we have $Th = \gamma Tg$ for some non-zero scalar γ. Equations (52) show that $\alpha(\gamma g) = \alpha(g)$ for all scalars γ. By scaling h appropriately, we may assume that $Th = -Tg$. Substituting into Equation (54) yields $\alpha(g) = \alpha(h)$, giving the result. $\square$

Proposition 7.10. *Let $\mathcal{H}$ be a Hilbert space. If $\mathcal{S}$ is a subspace of $\mathcal{B}(\mathcal{H})$, then $(\mathrm{Ref}\,\mathcal{S})_\perp$ is the closed linear span of the rank 1 operators it contains. Consequently, a weak*-closed subspace of $\mathcal{B}(\mathcal{H})$ is reflexive if and only if its preannihilator is the closed linear span of the rank 1 operators it contains.*

Proof. We first show that $\mathcal{S}_\perp$ and $(\operatorname{Ref}\mathcal{S})_\perp$ have the same rank 1 operators. For $g, h \in \mathcal{H}$, we have $g \otimes h^* \in \mathcal{S}_\perp$ if and only if $h \in [\mathcal{S}g]^\perp$. However, by definition of $\operatorname{Ref}\mathcal{S}$, we have $[\mathcal{S}g] = [\operatorname{Ref}\mathcal{S}g]$, and so $g \otimes h^* \in \mathcal{S}_\perp$ if and only if $g \otimes h^* \in (\operatorname{Ref}\mathcal{S})_\perp$.

Let $\mathcal{X}$ be the closed linear span of the rank 1 operators in $\mathcal{S}_\perp$. From the above argument, $\mathcal{X} \subseteq (\operatorname{Ref}\mathcal{S})_\perp$. To show the other containment, it suffices by the Hahn-Banach Theorem to show that $\mathcal{X}^\perp \subseteq ((\operatorname{Ref}\mathcal{S})_\perp)^\perp = \operatorname{Ref}\mathcal{S}$, where the last equality follows from the WOT-closure and hence weak*-closure of $\operatorname{Ref}\mathcal{S}$.

Let $T \in \mathcal{X}^\perp$ and $g \in \mathcal{H}$. If $h \in [\mathcal{S}g]^\perp$, then $g \otimes h^* \in \mathcal{S}_\perp$ and hence $g \otimes h^* \in \mathcal{X}$. Hence $0 = \operatorname{tr} T(g \otimes h^*) = \langle Tg, h\rangle$. Since h was arbitrary in $[\mathcal{S}g]^\perp$, $Tg \in [\mathcal{S}g]$ and so $T \in \operatorname{Ref}\mathcal{S}$.

For the second assertion, note that if $\mathcal{S}$ is weak*-closed and $\mathcal{S}_\perp = (\operatorname{Ref}\mathcal{S})_\perp$, then by weak*-closure of $\mathcal{S}$, $\mathcal{S} = (\mathcal{S}_\perp)^\perp = ((\operatorname{Ref}\mathcal{S})_\perp)^\perp = \operatorname{Ref}\mathcal{S}$. $\square$

Hyperreflexivity

Our extensions of certain notions to an almost invariant setting have thus far been somewhat unsatisfactory. Nevertheless, we continue to study the notion of hyperreflexivity. It is possible to try to extend the definition of hyperreflexivity to the lattice of almost invariant subspaces, however it is unclear how well this will hold up.

Definition 8.1. Let $\mathcal{H}$ be a Hilbert space. If $\mathcal{S}$ is a subspace of $\mathcal{B}(\mathcal{H})$, we define the quantity

$$\alpha(T, \mathcal{S}) = \sup\left\{\|Q^\perp TP\| \colon P, Q \text{ are projections and } Q^\perp \mathcal{S} P = \{0\}\right\}. \tag{55}$$

Lemma 8.2. *Let $\mathcal{H}$ be a Hilbert space. If $\mathcal{S}$ is a subspace of $\mathcal{B}(\mathcal{H})$, and P, Q are projections then $Q^\perp \mathcal{S} P = \{0\}$ if and only if $g \otimes h^* \in \mathcal{S}_\perp$ whenever $g \in P$ and $h \in Q^\perp$.*

Proof. Suppose that $Q^\perp \mathcal{S} P = \{0\}$, then for each $g \in P$ and $h \in Q^\perp$, $\langle Ag, h\rangle = 0$ for each $A \in \mathcal{S}$. If $h = 0$, then $g \otimes h^* = 0 \in \mathcal{S}_\perp$. Otherwise we may extend

$h_0 = \frac{h}{\|h\|}$ to an orthonormal basis $\mathcal{E}$ of $\mathcal{H}$ and evaluate

$$\begin{aligned}
\operatorname{tr} A(g \otimes h^*) &= \operatorname{tr} Ag \otimes h^* \\
&= \sum_{e \in \mathcal{E}} \langle (Ag \otimes h^*)e, e\rangle \\
&= \sum_{e \in \mathcal{E}} \langle \langle e, h\rangle Ag, e\rangle \\
&= \langle \langle h_0, h\rangle Ag, h_0\rangle \\
&= \frac{1}{\|h\|^2}\langle Ag, h\rangle \\
&= 0.
\end{aligned} \tag{56}$$

Since this is true for each $A \in \mathcal{S}$, hence $g \otimes h^* \in \mathcal{S}_\perp$.

To show the converse, we essentially reverse the above argument: If $g \otimes h^* \in \mathcal{S}_\perp$, then for each $A \in \mathcal{S}$, we get $\operatorname{tr} A(g \otimes h^*) = \operatorname{tr} Ag \otimes h^* = 0$. If $h \neq 0$, then we may again extend $h_0 = \frac{h}{\|h\|}$ to an orthonormal basis $\mathcal{E}$ of $\mathcal{H}$. Equation (56) shows that $\langle Ag, h\rangle = 0$ for each $g \in P$ and $h \in Q^\perp$, hence $Q^\perp \mathcal{S} P = \{0\}$. □

We now prove some properties of the quantity $\alpha(T, \mathcal{S})$.

Proposition 8.3. *Let $\mathcal{H}$ be a Hilbert space. If $\mathcal{S}$ is a subspace of $\mathcal{B}(\mathcal{H})$, then for each $T \in \mathcal{B}(\mathcal{H})$*

a) $\alpha(T, \mathcal{S}) \leqslant \operatorname{dist}(T, \mathcal{S})$;

b) $\|T\|_{\mathcal{S}} = \alpha(T, \mathcal{S})$ *is a seminorm on* $\mathcal{B}(\mathcal{H})$;

c) $\alpha(T, \mathcal{S}) = \sup\left\{|\langle Tg, h\rangle| \colon g \otimes h^* \in \operatorname{ball} \mathcal{S}_\perp\right\}$;

d) $\alpha(T, \mathcal{S}) = 0$ *if and only if* $T \in \operatorname{Ref} \mathcal{S}$;

e) If $\mathcal{A}$ is a subalgebra of $\mathcal{B}(\mathcal{H})$ containing the identity, then

$$\alpha(T, \mathcal{A}) = \sup\left\{\|P^\perp T P\| \colon P \in \operatorname{Lat} \mathcal{A}\right\}. \tag{57}$$

Proof. a) If $A \in \mathcal{S}$ and P, Q are projections with $Q^\perp \mathcal{S} P = \{0\}$, then for each $T \in \mathcal{B}(\mathcal{H})$,

$$\|Q^\perp T P\| = \|Q^\perp (T - A) P\| \leqslant \|T - A\|. \tag{58}$$

Thus we see that $\alpha(T, \mathcal{S}) \leqslant \operatorname{dist}(T, A)$ for each $A \in \mathcal{S}$. The result follows.

b) This follows from $\|\cdot\|$ being a norm on $\mathcal{B}(\mathcal{H})$.

c) Let $\beta(T, \mathcal{S}) = \sup\left\{|\langle Tg, h\rangle| \colon g \otimes h^* \in \operatorname{ball} \mathcal{S}_\perp\right\}$. We note that if $\|g \otimes h^*\| = \|g\|\|h\| \leqslant 1$, then we may assume that $\|g\| = \|h\| \leqslant 1$. By Lemma 8.2, when $Q^\perp \mathcal{S} P = \{0\}$, then $g \otimes h^* \in \mathcal{S}_\perp$ for $g \in P$ and $h \in Q^\perp$. Hence

$$\|Q^\perp T P\| = \sup\left\{|\langle Tg, h\rangle| \colon g \in \operatorname{ball} P, h \in \operatorname{ball} Q^\perp\right\} \leqslant \beta(T, \mathcal{S}). \tag{59}$$

So $\alpha(T,\mathcal{S}) \leqslant \beta(T,\mathcal{S})$.

For the other inequality, if $g \otimes h^* \in \mathcal{S}_\perp$ with $\|g\| = \|h\| \leqslant 1$ and P, Q are the projections onto $\operatorname{span}\{g\}$ and $[\mathcal{S}g]$ respectively, then $Q^\perp \mathcal{S} P = \{0\}$. Also,

$$|\langle Tg, h\rangle| \leqslant \|Q^\perp T P\| \leqslant \alpha(T,\mathcal{S}). \tag{60}$$

Thus $\beta(T,\mathcal{S}) \leqslant \alpha(T,\mathcal{S})$.

d) By Part c), $\alpha(T,\mathcal{S}) = 0$ if and only if $\sup\left\{|\langle Tg,h\rangle| \colon g \otimes h^* \in \operatorname{ball}\mathcal{S}_\perp\right\} = 0$. By Proposition 7.10, since $(\operatorname{Ref}\mathcal{S})_\perp$ is the closed linear span of the rank 1 operators it contains, and $\mathcal{S}_\perp$ and $(\operatorname{Ref}\mathcal{S})_\perp$ share the same rank 1 operators, it follows that $\alpha(T,\mathcal{S}) = 0$ if and only if $T \in ((\operatorname{Ref}\mathcal{S})_\perp)^\perp = \operatorname{Ref}\mathcal{S}$.

e) Let $\delta(T,\mathcal{A}) = \sup\left\{\|P^\perp T P\| \colon P \in \operatorname{Lat}\mathcal{A}\right\}$. If $P \in \operatorname{Lat}\mathcal{A}$, then $P^\perp \mathcal{A} P = \{0\}$, so $\delta(T,\mathcal{A}) \leqslant \alpha(T,\mathcal{A})$.

Suppose P, Q are projections satisfying $Q^\perp \mathcal{A} P = \{0\}$. Let $\widehat{P}$ be the projection onto $[\mathcal{A}P\mathcal{H}]$. $Q^\perp \widehat{P} = 0$ hence $\widehat{P} \subseteq Q$. Since $\mathcal{A}$ is an algebra, it follows that $\widehat{P} \in \operatorname{Lat}\mathcal{A}$. Since $\mathcal{A}$ contains the identity, $P \subseteq \widehat{P}$.

For any $h \in \mathcal{H}$,

$$\begin{aligned}
\|Q^\perp T P h\|^2 &= \langle Q^\perp T P h, Q^\perp T P h\rangle \\
&= \langle Q^\perp T P h, T P h\rangle \\
&\leqslant \langle \widehat{P}^\perp T P h, T P h\rangle \\
&= \langle \widehat{P}^\perp T \widehat{P} P h, T \widehat{P} P h\rangle \\
&= \|\widehat{P}^\perp T \widehat{P} P h\|^2 \\
&\leqslant \|\widehat{P}^\perp T \widehat{P}\|^2 \|P h\|^2.
\end{aligned} \tag{61}$$

This shows that $\|\widehat{P}^\perp T \widehat{P}\| \geqslant \|Q^\perp T P\|$ and so $\delta(T,\mathcal{A}) \geqslant \alpha(T,\mathcal{A})$.

□

Definition 8.4. Let $\mathcal{H}$ be a Hilbert space. A subspace $\mathcal{S}$ of $\mathcal{B}(\mathcal{H})$ is called **hyperreflexive** if there exists a constant c such that for each $T \in \mathcal{B}(\mathcal{H})$,

$$\operatorname{dist}(T,\mathcal{S}) \leqslant c\alpha(T,\mathcal{S}). \tag{62}$$

To make the importance of the constant c in Equation (62) explicit, in this case, we will say that $\mathcal{S}$ is **hyperreflexive with constant** c. We also let $\kappa(\mathcal{S})$ be the infimum of all such constants c. Equation (62) and Part a) of Proposition 8.3 together tell us that

$$\alpha(T,\mathcal{S}) \leqslant \operatorname{dist}(T,\mathcal{S}) \leqslant c\alpha(T,\mathcal{S}). \tag{63}$$

Hence $\alpha(\cdot,S)$ and $\operatorname{dist}(\cdot,S)$ are equivalent seminorms on $\mathcal{B}(H)$.

As in the definition of reflexivity, we say an operator T is hyperreflexive with constant c if the WOT-closed algebra generated by T and the identity is hyperreflexive with constant c.

Example 8.5. Consider the diagonal operators $\mathcal{D}_n$ with respect to a fixed orthonormal basis in an n-dimensional Hilbert space $\mathcal{H}$.

It is clear that $\mathcal{D}_1$ is hyperreflexive with constant 1 since in this case, every operator is diagonal.

If $\mathcal{H}$ is 2-dimensional, we can calculate the values $\alpha(T, \mathcal{D}_2)$ and $\operatorname{dist}(T, \mathcal{D}_2)$ for arbitrary $T \in \mathcal{B}(\mathcal{H})$ relatively easily.

Consider an arbitrary 2×2 matrix

$$T = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Quick calculations using the formula from Part e) of Proposition 8.3 will show that

$$\alpha(T, \mathcal{D}_2) = \max \left\{ \left\| \begin{bmatrix} 0 & b \\ 0 & 0 \end{bmatrix} \right\|, \left\| \begin{bmatrix} 0 & 0 \\ c & 0 \end{bmatrix} \right\| \right\} = \max \{|b|, |c|\} \tag{64}$$

and that

$$\operatorname{dist}(T, \mathcal{D}_2) \leqslant \left\| \begin{bmatrix} 0 & b \\ c & 0 \end{bmatrix} \right\| = \max \{|b|, |c|\} = \alpha(T, \mathcal{D}_2). \tag{65}$$

Equations (64) and (65) show that in the 2-dimensional case, the diagonal operators $\mathcal{D}_2$ are hyperreflexive with constant 1.

Remark 8.6. Calculating $\alpha(T, \mathcal{D}_n)$ and $\operatorname{dist}(T, \mathcal{D}_n)$ becomes much more complicated in higher dimensions. For example, in the 3-dimensional case, for an arbitrary matrix

$$T = \begin{bmatrix} a & b & c \\ d & e & f \\ g & h & i \end{bmatrix},$$

one can use the same method as before and calculate that

$$\begin{aligned} \alpha(T, \mathcal{D}_3) = \max\{&\sqrt{|d|^2 + |g|^2}, \sqrt{|b|^2 + |h|^2}, \sqrt{|c|^2 + |f|^2}, \\ &\sqrt{|g|^2 + |h|^2}, \sqrt{|d|^2 + |f|^2}, \sqrt{|b|^2 + |c|^2}\}. \end{aligned} \tag{66}$$

Calculating $\operatorname{dist}(T, \mathcal{D}_3)$ in full generality is difficult, so one may use methods such as Gershgorin's Circle Theorem to achieve upper bounds for this value.

In light of the previous remark, we are still able to show that $\mathcal{D}_n$ is hyperreflexive for any finite n. We will be able to produce an upper bound (though not necessarily a good one) on $\kappa(\mathcal{D}_n)$. The methods that follow come from work by Bessonov et al. [13] and by Kliś and Ptak [14].

Lemma 8.7. *Let $\mathcal{H}$ be a Hilbert space. If $\mathcal{S}$ is a subspace of $\mathcal{B}(\mathcal{H})$ and $T \in \mathcal{B}(\mathcal{H})$, then*

$$\alpha(T, \mathcal{S}) = \sup_{\|x\| \leqslant 1} \inf_{A \in \mathcal{S}} \left\{ \|(T - A)x\| \right\}. \tag{67}$$

Proof. Note that $\inf_{A \in \mathcal{S}} \left\{ \|(T - S)x\| \right\} = \|Q^{\perp} T x\|$ where Q is the projection onto $[\mathcal{S}x]$. If $\|x\| \leqslant 1$ and we pick P to be the projection onto $\operatorname{span}\{x\}$, we see that $\|Q^{\perp} T x\| \leqslant \|Q^{\perp} T P\|$. Since $Q^{\perp} \mathcal{S} P = \{0\}$, we have $\alpha(T, \mathcal{S}) \geqslant \sup_{\|x\| \leqslant 1} \inf_{A \in \mathcal{S}} \left\{ \|(T - A)x\| \right\}$.

If P, Q are projections with $Q^{\perp} \mathcal{S} P = \{0\}$ and $\epsilon > 0$, we may pick $x \in P\mathcal{H}$ such that $\|x\| = 1$ and $\|Q^{\perp} T x\| \geqslant \|Q^{\perp} T P\| - \epsilon$. For $A \in \mathcal{S}$,

$$\|(T - A)x\| \geqslant \operatorname{dist}(Tx, Q\mathcal{H}) = \|Q^{\perp} T x\| \geqslant \|Q^{\perp} T P\| - \epsilon. \tag{68}$$

Since ϵ was arbitrary, we get the result by taking the supremum over all projections P, Q with $Q^{\perp} \mathcal{S} P = \{0\}$. $\square$

We now prove a hyperreflexive version of Proposition 7.5, which will show that the direct sum of hyperreflexive subspaces is still hyperreflexive.

Proposition 8.8. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces. Let $\mathcal{S}_1$ and $\mathcal{S}_2$ be subspaces of $\mathcal{B}(\mathcal{H}_1)$ and $\mathcal{B}(\mathcal{H}_2)$ respectively. If $T_1 \in \mathcal{B}(\mathcal{H}_1)$ and $T_2 \in \mathcal{B}(\mathcal{H}_2)$, then*

a) $\max \left\{ \operatorname{dist}(T_1, \mathcal{S}_1), \operatorname{dist}(T_2, \mathcal{S}_2) \right\} = \operatorname{dist}(T_1 \oplus T_2, \mathcal{S}_1 \oplus \mathcal{S}_2)$;

b) $\max \left\{ \alpha(T_1, \mathcal{S}_1), \alpha(T_2, \mathcal{S}_2) \right\} \leqslant \alpha(T_1 \oplus T_2, \mathcal{S}_1 \oplus \mathcal{S}_2) \leqslant \alpha(T_1, \mathcal{S}_1) + \alpha(T_2, \mathcal{S}_2)$;

c) $\mathcal{S}_1 \oplus \mathcal{S}_2$ *is hyperreflexive if and only if both* $\mathcal{S}_1$ *and* $\mathcal{S}_2$ *are, and*

$$\max \left\{ \kappa(\mathcal{S}_1), \kappa(\mathcal{S}_2 \right\}) \leqslant \kappa(\mathcal{S}_1 \oplus \mathcal{S}_2) \leqslant 1 + 2 \max \left\{ \kappa(\mathcal{S}_1), \kappa(\mathcal{S}_2) \right\}. \tag{69}$$

Proof. a) We compute that

$$\begin{aligned} &\operatorname{dist}(T_1 \oplus T_2, \mathcal{S}_1 \oplus \mathcal{S}_2)^2 \\ =& \inf_{\substack{A_1 \in \mathcal{S}_1 \\ A_2 \in \mathcal{S}_2}} \sup_{\|x_1 \oplus x_2\| \leqslant 1} \left\{ \|(T_1 - A_1)x_1\|^2 + \|(T_2 - A_2)x_2\|^2 \right\} \\ \geqslant& \inf_{A_1 \in \mathcal{S}_1} \sup_{\|x_1\| \leqslant 1} \left\{ \|(T_1 - A_1)x_1\|^2 \right\} \\ =& \operatorname{dist}(T_1, \mathcal{S}_1)^2. \end{aligned} \tag{70}$$

The same will hold if we had used the index 2 instead of 1 in the last two lines of Equation (70).

We also see that

$$
\begin{aligned}
&\operatorname{dist}(T_1 \oplus T_2, \mathcal{S}_1 \oplus \mathcal{S}_2)^2 \\
=& \inf_{\substack{A_1 \in \mathcal{S}_1 \\ A_2 \in \mathcal{S}_2}} \sup_{\|x_1 \oplus x_2\| \leqslant 1} \left\{ \|(T_1 - A_1)x_1\|^2 + \|(T_2 - A_2)x_2\|^2 \right\} \\
\leqslant& \inf_{\substack{A_1 \in \mathcal{S}_1 \\ A_2 \in \mathcal{S}_2}} \sup_{\substack{\|x_1\| \leqslant 1 \\ \|x_2\| \leqslant 1}} \max \left\{ \|(T_1 - A_1)x_1\|^2, \|(T_2 - A_2)x_2\|^2 \right\} \\
\leqslant& \max \left\{ \operatorname{dist}(T_1, \mathcal{S}_1)^2, \operatorname{dist}(T_2, \mathcal{S}_2)^2 \right\}.
\end{aligned} \tag{71}
$$

b) Using Lemma 8.7, we get

$$
\begin{aligned}
\alpha(T_1, \mathcal{S}_1) &= \sup_{\|x_1\| \leqslant 1} \inf_{A_1 \in \mathcal{S}_1} \left\{ \|(T_1 - A_1)x_1\| \right\} \\
&= \sup_{\|x_1\| \leqslant 1} \inf_{\substack{A_1 \in \mathcal{S}_1 \\ A_2 \in \mathcal{S}_2}} \left\{ \|(T_1 \oplus T_2 - A_1 \oplus A_2)(x_1 \oplus 0)\| \right\} \\
&\leqslant \alpha(T_1 \oplus T_2, \mathcal{S}_1 \oplus \mathcal{S}_2) \\
&\leqslant \alpha(T_1 \oplus 0, \mathcal{S}_1 \oplus \mathcal{S}_2) + \alpha(0 \oplus T_2, \mathcal{S}_1 \oplus \mathcal{S}_2).
\end{aligned} \tag{72}
$$

The last inequality follows as by Part b) of Proposition 8.3, $\alpha(\cdot, \mathcal{S}_1 \oplus \mathcal{S}_2)$ is a seminorm. We also have

$$
\begin{aligned}
&\alpha(T_1 \oplus 0, \mathcal{S}_1 \oplus \mathcal{S}_2) \\
=& \sup_{\|x_1 \oplus x_2\| \leqslant 1} \inf_{\substack{A_1 \in \mathcal{S}_1 \\ A_2 \in \mathcal{S}_2}} \left\{ \|(T_1 - A_1)x_1\|^2 + \|A_2 x_2\|^2 \right\} \\
=& \sup_{\|x_1\| \leqslant 1} \inf_{A_1 \in \mathcal{S}_1} \left\{ \|(T_1 - A_1)x_1\|^2 \right\} \\
=& \alpha(T_1, \mathcal{S}_1)^2.
\end{aligned} \tag{73}
$$

Again, swapping the indices 1 and 2 will not change the validity of Equation (73), so the result follows.

c) Suppose first that $\mathcal{S}_1 \oplus \mathcal{S}_2$ is hyperreflexive. Then

$$
\begin{aligned}
\operatorname{dist}(T_1, \mathcal{S}_1) &= \operatorname{dist}(T_1 \oplus 0, \mathcal{S}_1 \oplus \mathcal{S}_2) \\
&\leqslant \kappa(\mathcal{S}_1 \oplus \mathcal{S}_2) \alpha(T_1 \oplus 0, \mathcal{S}_1 \oplus \mathcal{S}_2) \\
&= \kappa(\mathcal{S}_1) \oplus \mathcal{S}_2) \alpha(T_1, \mathcal{S}_1).
\end{aligned} \tag{74}
$$

Similarly, we conclude that $\operatorname{dist}(T_2, \mathcal{S}_2) \leqslant \kappa(\mathcal{S}_1 \oplus \mathcal{S}_2) \alpha(T_2, \mathcal{S}_2)$. This gives $\max \left\{ \kappa(\mathcal{S}_1), \kappa(\mathcal{S}_2) \right\} \leqslant \kappa(\mathcal{S}_1 \oplus \mathcal{S}_2)$.

In order to continue, we first need to prove some other preliminary results. We will return to this proof after doing so.

□

Proposition 8.9. *Let $\mathcal{H}$ be a Hilbert space. If $\mathcal{S}$ is a hyperreflexive subspace of $\mathcal{B}(\mathcal{H})$, then $\mathcal{S}$ is reflexive. In particular, every hyperreflexive subspace is WOT-closed.*

Proof. If $T \in \operatorname{Ref}\mathcal{S}$, then by Part d) of Proposition 8.3, $\alpha(T,\mathcal{S}) = 0$. Hyperreflexivity of $\mathcal{S}$ implies that $\operatorname{dist}(T,\mathcal{S}) \leqslant c\alpha(T,\mathcal{S}) = 0$. Hence $T \in \mathcal{S}$ and $\mathcal{S}$ is reflexive.

WOT-closure of $\mathcal{S}$ follows from Part a) of Proposition 7.2. □

Lemma 8.10. *Let $\mathcal{H}$ be a Hilbert space. Let $\mathcal{S}$ be a hyperreflexive subspace of $\mathcal{B}(\mathcal{H})$ with constant c_1. Let $\mathcal{L}$ be a subspace of $\mathcal{S}$ such that for any $A \in \mathcal{S}$, the following holds:*

$$\operatorname{dist}(A,\mathcal{L}) \leqslant c_2\alpha(A,\mathcal{L}) \tag{75}$$

for some constant c_2. Then $\mathcal{L}$ is hyperreflexive and $\kappa(\mathcal{L}) \leqslant c_1 + c_2 + c_1c_2$.

Proof. Let $T \in \mathcal{B}(\mathcal{H})$. For each $A \in \mathcal{S}$, we get $\operatorname{dist}(T,\mathcal{L}) \leqslant \|T - A\| + \operatorname{dist}(A,\mathcal{L})$. Since $\mathcal{S}$ is hyperreflexive with constant c_1, by Proposition 8.9, $\mathcal{S}$ is WOT-closed, hence weak*-closed. This shows that the distance between T and $\mathcal{S}$ is attained by some $A \in \mathcal{S}$. Hyperreflexivity of $\mathcal{S}$ and $\mathcal{L} \subseteq \mathcal{S}$ then imply that

$$\|T - A\| = \operatorname{dist}(T,\mathcal{S}) \leqslant c_1\alpha(T,\mathcal{S}) \leqslant c_1\alpha(T,\mathcal{L}). \tag{76}$$

Combining the above yields

$$\begin{aligned}
\operatorname{dist}(T,\mathcal{L}) &\leqslant \|T - A\| + \operatorname{dist}(A,\mathcal{L}) \\
&\leqslant c_1\alpha(T,\mathcal{L}) + c_2\alpha(A,\mathcal{L}) \\
&\leqslant c_1\alpha(T,\mathcal{L}) + c_2\alpha(A - T,\mathcal{L}) + c_2\alpha(T,\mathcal{L}) \\
&\leqslant (c_1 + c_2)\alpha(T,\mathcal{L}) + c_2\operatorname{dist}(A - T,\mathcal{L}) \\
&\leqslant (c_1 + c_2)\alpha(T,\mathcal{L}) + c_2\|A - T\| \\
&\leqslant (c_1 + c_2)\alpha(T,\mathcal{L}) + c_1c_2\alpha(T,\mathcal{L}) \\
&= (c_1 + c_2 + c_1c_2)\alpha(T,\mathcal{L}).
\end{aligned} \tag{77}$$

□

Lemma 8.11. *Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be Hilbert spaces. Then $\mathcal{B}(\mathcal{H}_1) \oplus \mathcal{B}(\mathcal{H}_2)$ is a hyperreflexive subalgebra of $\mathcal{B}(\mathcal{H}_1 \oplus \mathcal{H}_2)$ with constant 1.*

With Lemmas 8.10 and 8.11, we are now able to complete the proof of Proposition 8.8.

Proof of Proposition 8.8 Part c). What remains is to show that if $\mathcal{S}_1$ and $\mathcal{S}_2$ are hyperreflexive, then so is $\mathcal{S}_1 \oplus \mathcal{S}_2$ and that $\kappa(\mathcal{S}_1 \oplus \mathcal{S}_2) \leqslant 1 + 2\max\{\kappa(\mathcal{S}_1), \kappa(\mathcal{S}_2)\}$.

Let $T_1 \oplus T_2 \in \mathcal{B}(\mathcal{H}_1) \oplus \mathcal{B}(\mathcal{H}_2)$ and suppose that $\mathcal{S}_1$ and $\mathcal{S}_2$ are hyperreflexive. From Part a), $\operatorname{dist}(T_1 \oplus T_2, \mathcal{S}_1 \oplus \mathcal{S}_2) = \max\left\{\operatorname{dist}(T_1, \mathcal{S}_1), \operatorname{dist}(T_2, \mathcal{S}_2)\right\}$. Hyperreflexivity gives,

$$\begin{aligned} \operatorname{dist}(T_1, \mathcal{S}_1) &\leqslant \kappa(\mathcal{S}_1)\alpha(T_1, \mathcal{S}_1) \\ \operatorname{dist}(T_2, \mathcal{S}_2) &\leqslant \kappa(\mathcal{S}_2)\alpha(T_2, \mathcal{S}_2). \end{aligned} \tag{78}$$

Combining the above, we get

$$\begin{aligned} \operatorname{dist}(T_1 \oplus T_2, \mathcal{S}_1 \oplus \mathcal{S}_2) &= \max\left\{\operatorname{dist}(T_1, \mathcal{S}_1), \operatorname{dist}(T_2, \mathcal{S}_2)\right\} \\ &\leqslant \max\left\{\kappa(\mathcal{S}_1)\alpha(T_1, \mathcal{S}_1), \kappa(\mathcal{S}_2)\alpha(T_2, \mathcal{S}_2)\right\} \\ &\leqslant \max\left\{\kappa(\mathcal{S}_1), \kappa(\mathcal{S}_2)\right\} \max\left\{\alpha(T_1, \mathcal{S}_1), \alpha(T_2, \mathcal{S}_2)\right\} \\ &\leqslant \max\left\{\kappa(\mathcal{S}_1), \kappa(\mathcal{S}_2)\right\} \alpha(T_1 \oplus T_2, \mathcal{S}_1 \oplus \mathcal{S}_2). \end{aligned} \tag{79}$$

We may now apply Lemmas 8.10 and 8.11 to see that $\mathcal{S}_1 \oplus \mathcal{S}_2$ is hyperreflexive and $\kappa(\mathcal{S}_1 \oplus \mathcal{S}_2) \leqslant 1 + 2\max\left\{\kappa(\mathcal{S}_1), \kappa(\mathcal{S}_2)\right\}$. $\square$

Example 8.12. With Proposition 8.8, we can show that the diagonal operators $\mathcal{D}_n$ with respect to a fixed orthonormal basis in finite dimensions are hyperreflexive. In particular, we may write the 3×3 diagonals $\mathcal{D}_3$ as the direct sum $\mathcal{D}_3 = \mathcal{D}_1 \oplus \mathcal{D}_2$. By Example 8.5, $\kappa(\mathcal{D}_1) = \kappa(\mathcal{D}_2) = 1$, and so $\kappa(\mathcal{D}_3) \leqslant 1 + 2(1) = 3$. Similarly, $\mathcal{D}_4 = \mathcal{D}_2 \oplus \mathcal{D}_2$ and so $\kappa(\mathcal{D}_4) \leqslant 1 + 2(1) = 3$ as well.

Remark 8.13. The bounds that we get from this decomposition are not great as they increase quite quickly as the dimension of the Hilbert space increases. It has been shown that there are tighter bounds for the constants of hyperreflexivity for each $\mathcal{D}_n$. Results by Davidson and Ordower [15] show that $\kappa(\mathcal{D}_3) = \sqrt{\frac{3}{2}}$ and that $\sqrt{\frac{4}{9}\sqrt{2} + 1} \leqslant \kappa(\mathcal{D}_4) \leqslant \frac{3}{2}$. Another result by Rosenoer [16] shows that $\kappa(\mathcal{D}_n) \leqslant 2$ for each $n \geqslant 1$.

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